Radboud Universiteit Nijmegen

Mathematical Foundations of Computer Science

On "pseudofull" subsets of the continuum

A discussion of the neglected paragraphs of Brouwers article *"Über Definitionsbereiche von Funktionen"*

Anna Oosting
s3053148

Supervisor:
Dr. W.H.M. Veldman

August 17, 2016

Abstract

In 1927 L.E.J. Brouwer wrote an article called *"Über Definitionsbereiche von Funktionen"* ("On the domains of definition of functions"). In this article he proves that, intuitionistically, every total function on the closed interval $[0, 1]$ is uniformly continuous. He then wonders if we can find a notion of a pseudofull domain (a pseudofull subset of $[0, 1]$) so that functions defined on a pseudofull domain are not necessarily (uniformly) continuous. A pseudofull domain will have to be very much 'alike' $[0, 1]$ and will have to be almost full in the measure theoretic sense. A classical mathematician would not see a difference between such a pseudofull domain and $[0, 1]$. In the fifth paragraph of his article he gives seven examples for possible pseudofull domains. In this thesis we investigated these examples and verify most of the properties Brouwer claims for them. For some it seems that Brouwer makes a mistake.

Contents

Introduction

In this chapter we will give an introduction to this thesis. The goal of this thesis is to discuss the results of the fifth paragraph of L.E.J. Brouwer's 1927 article *"Über Definitionsbereiche von Funktionen"* [4]. We will first give a historical introduction about this article. Then we will outline what we will discuss in this thesis. We will also introduce some notation that we will use and then discuss some intuitionistic background which will become very useful in this thesis.

A short historical introduction

In 1927 L.E.J. Brouwer wrote an article called *"Über Definitionsbereiche von Funktionen"* ("On the domains of definition of functions") [4]. In this article he proves that, intuitionistically, every total function on the closed interval $[0,1]$ is uniformly continuous (theorem 3). What is needed for this proof are his famous bar theorem and fan theorem, which are both discussed in this article. Both the bar theorem and the fan theorem are related with the way Brouwer treated the continuum. For more background information about the bar theorem and the fan theorem, see [9]. In paragraph 4 of the article he then wonders if we can find a notion of a pseudofull domain (a pseudofull subset of $[0,1]$) so that functions defined on a pseudofull domain are not necessarily (uniformly) continuous. A pseudofull domain will have to be very much 'alike' $[0,1]$ and will have to be almost full in the measure theoretic sense. A classical mathematician would not see a difference between such a pseudofull domain and $[0,1]$. In the fifth paragraph of his article he gives seven examples for possible pseudofull domains and discusses the properties of these examples. In this thesis we will investigate these examples and their properties.

Outline

In the first chapter we will introduce real numbers and their properties. Also, we will introduce the continuum and define a number of relations between subsets of the continuum, which have been introduced by Brouwer.
In the next chapter we will discuss continuity, which will involve the important continuity theorem and uniform continuity theorem. We will introduce the notion of a spread and the continuity principle, which are needed to prove the continuity theorem. Also we will introduce the notion of a fan, the notion of a bar, the fan theorem and the extended fan theorem, which are needed to prove the uniform continuity theorem.
In chapter 3 we will discuss intuitionistic measure theory. We will define what an almost full set is, when a function is measurable and when a set is measurable.
In the examples Brouwer gives in his article, he only discusses geometric types and in chapter 4 we will define what a geometric type is and discuss some intuitionistic mathematics on geometric types. This will be useful in the discussion of the examples of Brouwer.
In the final chapter we examine Brouwers examples and verify most of the properties that Brouwer claims for them. For some it seems that Brouwer makes a mistake.

Notation

We introduce a number of notations used in this thesis.
For every $n \in \mathbb{N}$, $\mathbb{N}^n$ is the set of all sequences of natural numbers of length n. So $\mathbb{N}^n := \{(a_0 \ldots, a_{n-1}) \mid \forall i < n[a_i \in \mathbb{N}]\}$.
$\mathbb{N}^*$ is the set of all finite sequences of natural numbers. So $\mathbb{N}^* = \bigcup_{n=0}^{\infty} \mathbb{N}^n$
$\mathbb{N}^{\mathbb{N}}$ is the set of all infinite sequences of natural numbers. So $\mathbb{N}^{\mathbb{N}} := \{(\alpha(0), \alpha(1), \alpha(2), \ldots) \mid \forall i \in \mathbb{N}[\alpha(i) \in \mathbb{N}]\}$.

Suppose $a \in \mathbb{N}^*$ and the length of a is n. Then, for every $i < n$, $\bar{a}i$ is the finite sequence consisting of the first i numbers of the sequence a. So, for every $i < n$, $\bar{a}i = (a_0, a_1, \ldots, a_{i-1})$.
Suppose $\alpha \in \mathbb{N}^{\mathbb{N}}$. Then, for every $i \in \mathbb{N}$, $\bar{\alpha}i$ are the first i numbers of the sequence α. So, for every $i \in \mathbb{N}$, $\bar{\alpha}i = (\alpha(0), \alpha(1), \ldots, \alpha(i-1))$.
Let $< > : \mathbb{N}^* \to \mathbb{N}$ be a fixed bijection between all finite sequences of natural numbers and the natural numbers. So, when $a_0, a_1, \ldots, a_n \in \mathbb{N}^*$ then $< a_0, a_1, \ldots, a_k > \in \mathbb{N}$. Also $< >$ is the natural number corresponding to the empty sequence.
Let $* : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function where for every $n, m \in \mathbb{N}$, $n * m$ corresponds to the concatenation of the sequences corresponding to n and m via the bijection $< >$.

Intuitionist background

We will now introduce some intuitionistic background, needed to understand this thesis. First we will introduce the natural numbers, the integers and the rational numbers. The real numbers will be introduced in chapter 1. Then we will introduce some intuitionistic logic.

The natural numbers, the integers and the rational numbers

In his dissertation *"On the Foundations of Mathematics"* [1] Brouwer first starts with a chapter called 'The construction of mathematics'. Here he first introduces the arithmetic of natural numbers, then he introduces the integers and the rational numbers.
Brouwer introduces the natural numbers as something very fundamental. The construction of the natural numbers is based on an observation of a move in time. Suppose we start now, we experience a past and a present. We give this past the number 1 and this present the number 2. But as soon as we number the past and the present, the present already becomes a new past and creates a new present. This new present will be the number 3 and continuing this way we create the natural numbers.
All kinds of rules for calculating with natural numbers, such as the commutative property, follow from the fundamental theorem of arithmetic as stated in [1]. This says that any fixed set of signs will give us the same natural number when we count it, independently of the order in which we count it. So, when we count a fixed set of signs we make a one-to-one correspondence with a sequence of natural numbers. And no matter in which order we count the signs, the sequence of natural numbers will stop at the same number.
From this follows, for example, $2 + 3 = 3 + 2$. By $2 + 3$ Brouwer means to first count to 2, but when counting on we let the elements after 2 have a one-to-one correspondence with the sequence $1, 2, 3$. So we get $1, 2, 3, 4, 5$ where $3, 4, 5$ corresponds to $1, 2, 3$. When we do a permutation we get $3, 4, 5, 1, 2$ but $3, 4, 5$ still corresponds to $1, 2, 3$ and $1, 2$ to $1, 2$, which is $3 + 2$. So the sequence $1, 2, 3, 4, 5$ obtained by $2 + 3$ can be counted in a different order and then corresponds one-to-one to what we get from $3 + 2$.
If we continue the sequence of natural numbers to the left we obtain $-1, -2, -3, \ldots$. Addition of integers is naturally defined by counting in two directions.
By a rational number we mean a pair of ordinal numbers (a,b) written in the form $\frac{a}{b}$. We define $\frac{-a}{b} = \frac{a}{-b}$ to make sure the denominator is always positive. We order the rational numbers by:

1. $\frac{a}{b} = \frac{c}{d}$ if and only if $a \times d = b \times c$, and
2. $\frac{a}{b} < \frac{c}{d}$ if and only if $a \times d < b \times c$, and
3. $\frac{a}{b} > \frac{c}{d}$ if and only if $a \times d > b \times c$

We define $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ and $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. Now commutativity, associativity and distributivity are clear.

Intuitionistic logic

We will now introduce some intuitionistic logic, but we will not prove any of it. See [6] for more background. First we will discuss some propositional logic and then some predicate logic. Suppose p, q are propositions, then:

(i) $\vdash p \implies \neg\neg p$

(ii) $\vdash (p \implies q) \implies (\neg q \implies \neg p)$

(iii) $\vdash \neg(p \lor q) \implies \neg p \land \neg q$

(iv) $\vdash \neg p \land \neg q \implies \neg(p \lor q)$

(v) $\vdash \neg p \lor \neg q \implies \neg(p \land q)$

(vi) $\vdash \neg\neg(p \lor \neg p)$

(vii) $\vdash \neg\neg(p \land q) \implies \neg\neg p \land \neg\neg q$

(viii) $\vdash \neg\neg p \land \neg\neg q \implies \neg\neg(p \land q)$

(ix) $\vdash \neg\neg p \lor \neg\neg q \implies \neg\neg(p \lor q)$

Note that the inverse implication of (i), (ii), (v) and (ix) do not hold.

Now suppose P is a predicate, then:

(i) $\vdash \forall x[P(x)] \implies \neg\exists x\neg[P(x)]$

(ii) $\vdash \exists x[P(x)] \implies \neg\forall x\neg[P(x)]$

(iii) $\vdash \forall x\neg[P(x)] \implies \neg\exists x[P(x)]$

(iv) $\vdash \neg\exists x[P(x)] \implies \forall x\neg[P(x)]$

(v) $\vdash \exists x\neg[P(x)] \implies \neg\forall x[P(x)]$

(vi) $\vdash \neg\neg\forall x\neg\neg[P(x)] \implies \neg\exists x\neg[P(x)]$

(vii) $\vdash \neg\neg\forall x[P(x)] \implies \forall x\neg\neg[P(x)]$

(viii) $\vdash \neg\neg\exists x[P(x)] \implies \neg\forall x\neg[P(x)]$

(ix) $\vdash \neg\forall x\neg[P(x)] \implies \neg\neg\exists x[P(x)]$

Note that the inverse implication of (i), (ii), (v), (vii) and (x) do not hold.

1 The continuum

In this chapter we will discuss the intuitionistic continuum. We will define the notion of a real number (1.1) and discuss some relations between real numbers (1.2). Then we will define a special real number (1.3), which will be very useful in the next chapters. Furthermore we will discuss how real numbers can relate to sets and how sets of real numbers can relate to each other.

1.1 Real numbers

In this section we will define real numbers. To define real numbers we need a fixed bijection $\lambda : \mathbb{N} \to \mathbb{Q} \times \mathbb{Q}$. Also we need two functions $P_0 : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ and $P_1 : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ with $P_0(q_0, q_1) = q_0$ and $P_1(q_0, q_1) = q_1$. We introduce the following abbreviations for all $n \in \mathbb{N}$: $n' = P_0(\lambda(n))$ and $n'' = P_1(\lambda(n))$.
Furthermore, $S = \{n \in \mathbb{N} \mid n' \leq n''\}$. So S is the set of (code numbers of) rational segments. For all $n, m \in \mathbb{N}$ we define $n \sqsubset m$ if and only if $m' < n' \leq n'' < m''$ or $n \sqsubseteq m$ if and only if $m' \leq n' \leq n'' \leq m''$. Lastly we define $l : \mathbb{N} \to \mathbb{Q}$, a length function for rational segments: for each n, $l(n) = n'' - n'$.
We will now define a real number as a function from $\mathbb{N}$ to $\mathbb{N}$.

Definition 1.1.1. *A* ***real number*** *x is a function $x : \mathbb{N} \to \mathbb{N}$ such that:*

(i) $\forall n \in \mathbb{N}\ [x(n) \in S]$ *(each $x(n)$ is a rational segment), and*

(ii) $\forall n \in \mathbb{N}\ [x(n+1) \sqsubset x(n)]$ *(each rational segment contains the next rational segment), and*

(iii) $\forall m \in \mathbb{N} \exists n \in \mathbb{N}\ [l(x(n)) \leq 2^{-m}]$ *(the rational segments get arbitrarily small).*

We now introduce $\mathbb{R} \subset \mathbb{N}^{\mathbb{N}}$, the set of all real numbers. For readability we define, for all real numbers x and for all $n \in \mathbb{N}$, $x'(n) = (x(n))'$ and $x''(n) = (x(n))''$.

We define a sequence of sets of 'canonical intervals', as follows

Definition 1.1.2. *For every $m \geq 1 \in \mathbb{N}$ we define:*

$$\lambda_m := \{n \in S \mid l(n)2^{m-1} = 1 \ \wedge\ n'2^m \in \mathbb{Z}\}$$

Definition 1.1.3. *A real numbers x is a* ***regular real number*** *if and only if for all $n \in \mathbb{N}$, $x(n) \in \lambda$, where $\lambda = \bigcup_{n \in \mathbb{N}} \lambda_n$.*

We define addition and multiplication on the real numbers.

Definition 1.1.4. *Suppose x and y are real numbers.*

(i) For each n, $(x+y)(n) := x(n) +_S y(n) := (x'(n) + y'(n), x''(n) + y''(n))$

(ii) For each n, define $M_n := \{x'(n)y'(n), x'(n)y''(n), x''(n)y'(n), x''(n)y''(n)\}$.
Define $(xy)(n) := x(n) \ \cdot_S\ y(n) := (\min(M_n), \max(M_n))$

It is easy to see that $x + y$ and xy are real numbers.

Definition 1.1.5. *For every real number x we define a real number $-x$. For each n,* $-x(n) := (-x'(n), -x''(n)) := (-(x''(n)), -(x'(n)))$.

Also, we define the distance between real numbers. Note that the distance between real numbers does not have to exist.

Definition 1.1.6. *Suppose x and y are real numbers. The distance between x and y is $|x-y| = \max(x-y, y-x)$.*

The following lemma will be useful.

Lemma 1.1.7. *Suppose x, y and z are real numbers and suppose $|x-y| \leq \frac{1}{n}$ and $|y-z| \leq \frac{1}{m}$, then $|x-z| \leq \frac{1}{n} + \frac{1}{m}$.*

Proof. Suppose x, y and z are real numbers and suppose $|x-y| \leq \frac{1}{n}$ and $|y-z| \leq \frac{1}{m}$. Then $x - y \leq \frac{1}{n}, y - x \leq \frac{1}{n}, y - z \leq \frac{1}{m}$ and $z - y \leq \frac{1}{m}$. So $x - z \leq x - (y - \frac{1}{m}) \leq \frac{1}{n} + \frac{1}{m}$. Also $z - x \leq z - (y - \frac{1}{n}) \leq \frac{1}{m} + \frac{1}{n}$. So $\max(z-x, x-z) \leq \frac{1}{n} + \frac{1}{m}$. □

In the next section we define a number of relations on real numbers.

1.2 Relations between real numbers

First we define the relations $<$ and $\leq$.

Definition 1.2.1. *Suppose x and y are real numbers. We define:*

(i) $x < y$ if and only if $\exists n \in \mathbb{N}[x''(n) < y'(n)]$

(ii) $x \leq y$ if and only if $\forall n \in \mathbb{N}[x'(n) \leq y''(n)]$

Next we define a number of (in)equality relations on real numbers. These notions are all introduced in [3] (1).

Definition 1.2.2. *Suppose x and y are real numbers. We define:*

*(i) x **coincides with** y (notation: $x \equiv y$) if and only if $\forall n \in \mathbb{N} \exists m \in \mathbb{N}[x(n) \sqsupseteq y(m)]$ and $\forall n \in \mathbb{N} \exists m \in \mathbb{N}[y(n) \sqsupseteq x(m)]$.* (Brouwer: Zusammenfallung)

*(ii) x **is apart from** y (notation: $x \# y$) if and only if $\exists n \in \mathbb{N} \exists m \in \mathbb{N}[x'(n) > y''(m) \ \vee \ x''(n) < y'(m)]$.* (Brouwer: Entfernung)

*(iii) x **deviates from** y (notation: $x \not\equiv y$) if and only if $\neg[x \equiv y]$.* (Brouwer: Abweichung)

There exist equivalent notions of apart and coincidence.

Lemma 1.2.3. *For all real numbers x, y,*

(i) $x \# y$ if and only if $\exists k \in \mathbb{N}[x'(k) > y''(k) \ \vee \ x''(k) < y'(k)]$, and

(ii) $x \equiv y$ if and only if $\forall n \in \mathbb{N} \ [x'(n) \leq y''(n) \ \wedge \ y'(n) \leq x''(n)]$.

Proof.

(i) Suppose x and y real numbers and $\exists n \in \mathbb{N} \exists m \in \mathbb{N}[x'(n) > y''(m) \ \vee \ x''(n) < y'(m)]$. Note that for all real numbers y: $\forall n \in \mathbb{N} \forall m \in \mathbb{N} \ [m > n \implies (y'(n) < y'(m) < y''(m) < y''(n))]$. Find $n, m \in \mathbb{N}$ such that $x'(n) > y''(m) \ \vee \ x''(n) < y'(m)$ and define $k = \max(n, m)$. Now $x'(k) > y''(k)$ or $x''(k) < y'(k)$.

(ii) Suppose x and y are real numbers and $x \equiv y$. We first note that for all real numbers y: $\forall n \in \mathbb{N} \forall m \in \mathbb{N} \ [y'(m) < y''(n)]$. Pick any $n \in \mathbb{N}$. Find $m \in \mathbb{N}$ such that $x(n) \sqsupseteq y(m)$. Then $x'(n) \leq y'(m) \leq y''(n)$. Also $y'(n) \leq y''(m) \leq x''(n)$. Now suppose $\forall n \in \mathbb{N} \ [x'(n) \leq y''(n) \ \wedge \ y'(n) \leq x''(n)]$ Pick any $n \in \mathbb{N}$. We distinguish two cases:

(1) There is an English translation of Brouwer's original article, see [8]. The terms used in this translation differ from the terms we use.

(a) $x'(n) \leq y'(n) \leq y''(n) \leq x''(n)$

(b) $y'(n) \leq x'(n) \leq y''(n) \leq x''(n)$

The cases $y'(n) \leq x'(n) \leq x''(n) \leq y''(n)$ and $x'(n) \leq y'(n) \leq x''(n) \leq y''(n)$ are of course equivalent, since x and y are arbitrarily chosen.
We first consider case 1. $x(n) \sqsupseteq y(n)$ so $\exists m \in \mathbb{N}[x(n) \sqsupseteq y(m)]$. Thus we need to find an $m \in \mathbb{N}$ such that $y(n) \sqsupseteq x(m)$. Since $y(n) \sqsubset y(n+1)$, there exists a $q > 0 \in \mathbb{Q}$ such that $y'(n+1) - y'(n) = q$. Find $m > n+1 \in \mathbb{N}$ such that $l(x(m)) \leq q$. Then $y'(n+1) < y'(m) \leq x''(m)$, so $y'(n) < x''(m) - q \leq x'(m)$, thus $y(n) \sqsupseteq x(m)$.
Now we consider case 2. There exists a $q > 0 \in \mathbb{Q}$ such that $y''(n) - y''(n+1) = q$. Find $m > n+1 \in \mathbb{N}$ such that $l(x(m)) \leq q$. Then $x'(m) \leq y''(m) < y''(n+1)$ so $x''(m) \leq x'(m) + q < y''(n)$ so $y(n) \sqsupseteq x(m)$. Also, there exists a $p > 0 \in \mathbb{Q}$ such that $x(n+1)' - x'(n) = p$. Find $m > n+1 \in \mathbb{N}$ such that $l(y(m)) \leq p$. Then $x'(n+1) < x'(m) \leq y''(m)$, so $x'(n) < y''(m) - p \leq y'(m)$, thus $x(n) \sqsupseteq y(m)$.

□

The following lemma shows that the definitions 1.2.2 *(i), (ii)* and *(iii)* are enough.

Lemma 1.2.4. *For all real numbers x, y,*

(i) if $\neg[x\#y]$ then $x \equiv y$, and

(ii) if $\neg\neg[x \equiv y]$ then $x \equiv y$, and

(iii) if $\neg\neg[x\#y]$ then $x \not\equiv y$.

Proof. Suppose x and y are real numbers.

(i) If $\neg[x\#y]$ then $\neg\exists n \in \mathbb{N}\exists m \in \mathbb{N}[x'(n) > y''(m) \vee\ x''(n) < y'(m)]$, so $\forall n \in \mathbb{N}\forall m \in \mathbb{N}[x'(n) \leq y''(m) \wedge\ y'(m) \leq x''(n)]$. Thus $\forall n \in \mathbb{N}[x'(n) \leq y''(n) \wedge\ y'(n) \leq x''(n)]$ so, by lemma 1.2.3, $x \equiv y$.

(ii) If $\neg\neg[x \equiv y]$ then, by lemma 1.2.3, $\neg\neg\forall n \in \mathbb{N}[x'(n) \leq y''(n) \wedge\ y'(n) \leq x''(n)]$. So $\forall n \in \mathbb{N}\neg\neg[x'(n) \leq y''(n) \wedge\ y'(n) \leq x''(n)]$, so $\forall n \in \mathbb{N}[\neg\neg(x'(n) \leq y''(n)) \wedge\ \neg\neg(y'(n) \leq x''(n))]$ and thus $\forall n \in \mathbb{N}[x'(n) \leq y''(n) \wedge\ y'(n) \leq x''(n)]$. Again, by lemma 1.2.3, $x \equiv y$.

(iii) Suppose $x\#y$. Find $n, m \in \mathbb{N}$ such that $x'(n) > y''(m)$ or $x''(n) < y'(m)$. Suppose $x \equiv y$. Assume $x'(n) > y''(m)$ then $\forall k \in \mathbb{N}[x'(n) > y'(k)]$, which is a contradiction. Now assume $x''(n) < y'(m)$ then $\forall k \in \mathbb{N}[x''(n) < y''(k)]$, which is also a contradiction. So if $x\#y$ then $\neg[x \equiv y]$ and thus if $\neg\neg[x \equiv y]$ then $\neg[x\#y]$. So if $\neg\neg[x\#y]$ then $\neg\neg\neg[x \equiv y]$ and so $x \not\equiv y$.

□

Suppose x and y are real numbers. Let $A \longrightarrow B$ be 'A implies B' and $A \overset{\lightning}{\longleftrightarrow} B$ be 'A and B contradict each other'. In the diagram below the connections between the relations of definition 1.2.2 are shown.

$$x \equiv y \underset{(1)}{\overset{\lightning}{\longleftrightarrow}} x \not\equiv y \underset{(2)}{\longleftarrow} x\#y$$

Figure 1: Relations between real numbers.

(1) is clear by the definition and for (2) see the proof of lemma 1.2.4 *(iii)*.

Lemma 1.2.5. *Suppose $x, y \in \mathbb{R}$. Then $x \# y \iff \exists m \in \mathbb{N}\ [|x-y| > \frac{1}{m}]$.*

Proof. Suppose $x\#y$. Then $\exists n \in \mathbb{N}\ \exists k \in \mathbb{N}\ [x'(n) >\ y''(k) \vee\ x''(n) <\ y'(k)]$. Suppose, without loss of generality, $\exists n \in \mathbb{N}\ \exists k \in \mathbb{N}$ such that $x'(n) >\ y''(k)$. Pick $n, k \in \mathbb{N}$ such that $x'(n) >\ y''(k)$. This means there exists a $m \in \mathbb{N}$ such that $x'(n) - y''(k) \geq \frac{1}{m}$. Thus $|x-y| \geq\ x'(n) - y''(k) > \frac{1}{m}$.
Suppose $\exists m \in \mathbb{N}\ [|x-y| > \frac{1}{m}]$. Find $m \in \mathbb{N}$ such that $|x-y| > \frac{1}{m}$. $|x-y| = \max(x-y, y-x)$. Suppose, without loss of generality, $|x-y| = x-y$. This means $x - y > \frac{1}{m}$. So, for some $n \in \mathbb{N}$ we have $(x-y)'(n) = (x+(-y))'(n) = x'(n) + (-y)'(n) = x'(n) + -(y''(n)) > \frac{1}{m}$ and thus $x'(n) >\ y''(n)$. □

Lemma 1.2.6. *Suppose $x, y \in \mathbb{R}$. Then $x \equiv y \iff \forall n \in \mathbb{R}\ |x-y| \leq \frac{1}{n}$.*

Proof. This follows directly from lemma 1.2.5 and lemma 1.2.4. □

1.3 A special real number

In this section we define a special real number r that will become useful in this thesis.

Definition 1.3.1. *Let $d_\pi : \mathbb{N} \to \{0, \ldots, 9\}$ be the decimal expansion of π, i.e. $\pi = \sum_{n=0}^{\infty} \frac{d_\pi(n)}{10^n}$. We take k_1 to be the least natural number n such that in the infinite sequence d_π a block of nine consecutive nines starts at position n. Note that we can not define k_1 as the name of a natural number, since we do not know if k_1 exists. What we can define are the three following predicates. For all $n \in \mathbb{N}$:*

- $n < k_1 \iff \forall m \leq n\ \neg[d_\pi(m) = d_\pi(m+1) = \cdots = d_\pi(m+8) = 9]$
- $n = k_1 \iff \big(d_\pi(n) = d_\pi(n+1) = \cdots = d_\pi(n+8) = 9 \wedge \forall m < n\ \neg[d_\pi(m) = d_\pi(m+1) = \cdots = d_\pi(m+8) = 9]\big)$
- $n > k_1 \iff \exists m < n\ [d_\pi(m) = d_\pi(m+1) = \cdots = d_\pi(m+8) = 9]$

We define the real number r to be the sequence $r(0), r(1), r(2), \ldots$ of rational intervals with:

$$r(n) = \begin{cases} ((-1)^{k_1} \sum_{i=k_1+2}^{n} \frac{1}{2^i}, (-1)^{k_1} \frac{1}{2^{k_1}} - \sum_{i=k_1+2}^{n} \frac{1}{2^i}) & \text{if } n \geq k_1 \text{ and } k_1 \text{ is even} \\ ((-1)^{k_1} \frac{1}{2^{k_1}} + \sum_{i=k_1+2}^{n} \frac{1}{2^i}, (-1)^{k_1} \sum_{i=k_1+2}^{n} \frac{1}{2^i}) & \text{if } n \geq k_1 \text{ and } k_1 \text{ is odd} \\ (\frac{-1}{2^n}, \frac{1}{2^n}) & \text{if } n < k_1 \end{cases}$$

So, if $\neg\exists n \in \mathbb{N}[n = k_1]$, then $r = 0$, if $\exists n \in \mathbb{N}[2n = k_1]$, then $r = \frac{1}{2^{k_1+1}}$ and if $\exists n \in \mathbb{N}[2n+1 = k_1]$, then $r = -\frac{1}{2^{k_1+1}}$.

We will discuss some properties of r. First of all, we can not prove that r is rational nor that r is irrational. Anyone who says 'r is rational' claims $\exists q \in \mathbb{Q}\ [q \equiv r]$. For all $q \in \mathbb{Q}$ we know $q < 0$, $q = 0$ or $q > 0$. Suppose $q < 0$ or $q > 0$, then $\exists n \in \mathbb{N}[n = k_1]$. Suppose $q = 0$, then $\neg\exists n \in \mathbb{N}[n = k_1]$. However, we do not have a proof of $\exists n \in \mathbb{N}[n = k_1]$ or of $\neg\exists n \in \mathbb{N}[n = k_1]$ so we do not have a proof of 'r is rational' and we do not have a proof of 'r is irrational'.
The following lemma shows that we can prove $\neg\neg$(r is rational), i.e. r is not irrational.

Lemma 1.3.2. $\neg\neg$(*r is rational*)

Proof. We distinguish two cases, namely: $\exists n \in \mathbb{N}\ [n = k_1]$ and: $\neg\exists n \in \mathbb{N}\ [n = k_1]$. If $\exists n \in \mathbb{N}\ [n = k_1]$ then $\exists n \in \mathbb{N}[2n = k_1]$ or $\exists n \in \mathbb{N}[2n + 1 = k_1]$. Suppose $\exists n \in \mathbb{N}[2n = k_1]$, then $r = \frac{-1}{2^{k_1+1}}$ and thus r is rational. Suppose $\exists n \in \mathbb{N}[2n + 1 = k_1]$, then $r = \frac{1}{2^{k_1+1}}$ and thus r is rational. So if $\exists n \in \mathbb{N}\ [n = k_1]$ then r is rational. If $\neg\exists n \in \mathbb{N}[n = k_1]$ then $r = 0$ and thus r is rational.
This gives us: if $(\exists n \in \mathbb{N}[n = k_1] \vee \neg\exists n \in \mathbb{N}[n = k_1])$ then r is rational. So if r is irrational then $\neg(\exists n \in \mathbb{N}[n = k_1] \vee \neg\exists n \in \mathbb{N}[n = k_1])$ and thus if $\neg\neg(\exists n \in \mathbb{N}[n = k_1] \vee \neg\exists n \in \mathbb{N}[n = k_1])$ then $\neg\neg(r$ is rational). Note that, for all propositions A: $\neg\neg(A \vee \neg A)$. So $\neg\neg(\exists n \in \mathbb{N}[n = k_1] \vee \neg\exists n \in \mathbb{N}[n = k_1])$ so $\neg\neg(r$ is rational). □

We can not prove (by the same argument) that r is negative, positive or zero. Anyone who says 'r is positive', claims $\exists n \in \mathbb{N}[n = k_1]$ and k_1 is even. Anyone who says 'r is negative', claims $\exists n \in \mathbb{N}[n = k_1]$ and k_1 is odd and anyone who says 'r is zero', claims $\neg\exists n \in \mathbb{N}[n = k_1]$.

1.4 Relations between real numbers and sets

In this section we will discuss how real numbers relate to sets. These notions are all introduced in [3].

Definition 1.4.1. *Suppose x is a real number and Y is a set of real numbers. We define:*

(i) x **is a member of** Y *(notation: $x \in_0 Y$) if and only if $\exists y \in Y[x \equiv y]$.* (Brouwer: Einhüllung)

(ii) x **is apart from** Y *(notation: $x \# Y$) if and only if $\forall y \in Y[x \# y]$.* (Brouwer: Entfernung)

(iii) x **is not a member of** Y *if and only if $\neg[x \in_0 Y]$.* (Brouwer: Abweichung)

(iv) x **seems to be a member of** Y *if and only if $\neg\neg[x \in_0 Y]$.* (Brouwer: Anschließung)

(v) x **is not apart from** Y *if and only if $\neg[x \# Y]$.* (Brouwer: Anlehnung)

(vi) x **seems to be apart from** Y *if and only if $\neg\neg[x \# Y]$.* (Brouwer: Abtrennung)

Again we have a diagram, written below, which shows us the connections between the above relations.

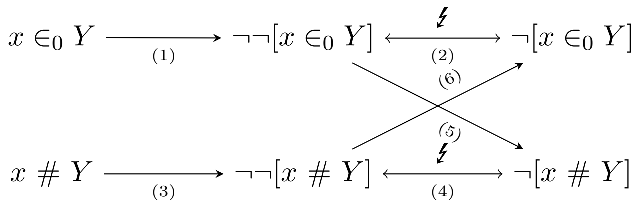

Figure 2: Relations between real numbers and sets.

Implications (1) and (3) and contradictions (2) and (4) are clear.

(5) Suppose x is a real number and Y a set such that $\neg\neg[x \in_0 Y]$. Then $\neg\neg[\exists y \in Y[x \equiv y]]$, so $\neg\forall y \in Y[x \not\equiv y]$. Now suppose $\forall y \in Y[x\#y]$ then, by figure 1, $\forall y \in Y[x \not\equiv y]$, which is a contradiction. So $\neg[x\#Y]$.

(6) Suppose x is a real number and Y a set such that $x\neg\neg[x \# Y]$. Then $\neg\neg[\forall y \in Y[x\#y]]$, so $\neg\exists y \in Y\neg[x\#y]$. This gives, by lemma 1.2.4, $\neg\exists y \in Y[x \equiv y]$ and thus $\neg[x \in_0 Y]$.

1.5 Relations between sets

In this section we discuss relations between sets of real numbers. These notions are all introduced in [3].

Definition 1.5.1. *Suppose X and Y are sets of real numbers. We define:*

(i) X **coincides with** Y *(notation: $X \equiv Y$) if and only if $\forall x \in X[x \in_0 Y]$ and $\forall y \in Y[y \in_0 X]$.* (Brouwer: Zusammenfallung)

(ii) X **deviates from** Y *(notation: $X \not\equiv Y$) if and only if $\exists x \in X \neg[x \in_0 Y]$ or $\exists y \in Y \neg[y \in_0 X]$.* (Brouwer: Abweichung)

(iii) X **is apart from** Y *(notation: $X \# Y$) if and only if $\exists x \in X[x \# Y]$ or $\exists y \in Y[y \# X]$.* (Brouwer: Entfernung)

(iv) X **does not coincide with** Y *if and only if $\neg[X \equiv Y]$.* (Brouwer: Loswindung)

(v) X **seems to coincide with** Y *if and only if $\neg\neg[X \equiv Y]$.* (Brouwer: Verflechtung)

(vi) X **does not deviate from** Y *if and only if $\neg[X \not\equiv Y]$.* (Brouwer: Kongruenz)

(vii) X **seems to deviate from** Y *if and only if $\neg\neg[X \not\equiv Y]$.* (Brouwer: Loslösung)

(viii) X **is not apart from** Y *if and only if $\neg[X \# Y]$.* (Brouwer: Übereinstimmung)

(ix) X **seems to be apart from** Y *if and only if $\neg\neg[X \# Y]$.* (Brouwer: Absonderung)

Again we have a diagram, written below, which shows us the connections between the above relations.

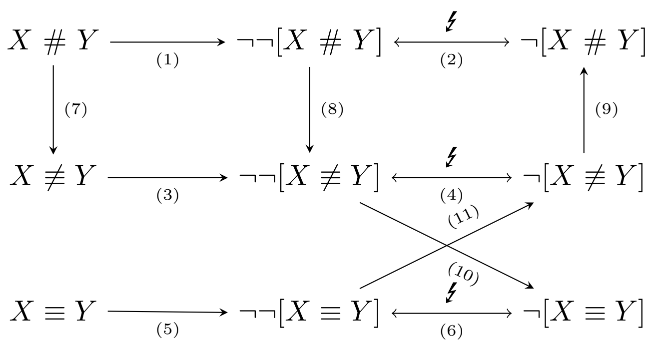

Figure 3: Relations between sets.

The implications (1), (3) and (5) are clear, as are the contradiction (2), (4) and (6).

(7) Suppose X and Y are sets such that $X \# Y$. Then $\exists x \in X\ \forall y \in Y[x \# y]$ or $\exists y \in Y\ \forall x \in X[y \# x]$. Assume $\exists x \in X \forall y \in Y[x \# y]$. Then, by figure 1, $\exists x \in X \forall y \in Y[x \not\equiv y]$ and thus $\exists x \in X \neg\exists y \in Y[x \equiv y]$, so $\exists x \in X \neg[x \in_0 Y]$, so $X \not\equiv Y$. The other case is similar.

(8) This implication follows directly from implication (7).

(9) Suppose X and Y are sets such that $\neg[X \not\equiv Y]$. Now suppose $X \# Y$, then by (1) and (8) we have $\neg\neg[X \not\equiv Y]$, which is a contradiction. So $\neg[X \# Y]$.

(10) Suppose X and Y are sets such that $\neg\neg[X \not\equiv Y]$. Then $\neg\neg[\exists x \in X \neg[x \in_0 Y] \vee \exists y \in Y \neg[y \in_0 X]]$, so $\neg[\neg\exists x \in X \neg[x \in_0 Y] \wedge \neg\exists y \in Y \neg[y \in_0 X]]$, so $\neg[\forall x \in X$ $neg \neg[x \in_0 Y] \wedge \forall y \in Y \neg\neg[y \in_0 X]]$. Now suppose $X \equiv Y$, then $\forall x \in X[x \in_0 Y] \ \wedge \ \forall y \in Y[y \in_0 X]$, which is a contradiction. So $\neg[X \equiv Y]$.

(11) Suppose X and Y are sets such that $\neg\neg[X \equiv Y]$. Now suppose $X \not\equiv Y$, by (3) and (10), this gives $\neg[X \equiv Y]$, which is a contradiction. So $\neg[X \not\equiv Y]$.

2 Continuity

In this chapter we will state that every real-valued total function is continuous and that every real-valued total function on $[0, 1]$ is uniformly continuous. To discuss these statements we will introduce two important notions, namely that of a spread (2.1) and of a fan (2.2). Also, the way Brouwer thought about the continuum led to the continuity principle (2.1), the fan theorem and the extended fan theorem (2.2) which are needed to prove the statements. We will not prove any of the theorems in 2.2 but refer to [9] and [10] for the proofs of these theorems and for further readings.

2.1 Spreads

In this section we will introduce spreads and the continuity principle. These two notions will help us prove that every real-valued total function is continuous.
First we will define what a spread is.

Definition 2.1.1. *A **spread-law** is a function $\sigma : \mathbb{N} \to \{0, 1\}$ such that:*

(i) $\sigma(<>) = 0$, *and*

(ii) $\forall s \in \mathbb{N}\big[\sigma(s) = 0 \iff \exists n \in \mathbb{N}[\sigma(s* < n >) = 0]\big]$.

You can think of a spread-law as an infinite tree with finite sequence at the nodes of the tree. The root of this tree is the empty sequence. By (ii) we know every node a has children which are the concatenation of a and a natural number n.

Definition 2.1.2. *Suppose $\alpha \in \mathbb{N}^{\mathbb{N}}$ and σ is a spread-law. We define $\alpha \in \sigma$ iff for every $n \in \mathbb{N}$, $\sigma(\bar{\alpha}n) = 0$. The set $\{\alpha \in \mathbb{N}^{\mathbb{N}} | \alpha \in \sigma\}$ is called a **spread**. We will also use σ to refer to this set.*

So, a spread is a set of infinite sequences which are accepted by a spread-law or the infinite sequences which are made by walking through the tree defined by the spread-law.
We now introduce some useful spreads.

Definition 2.1.3. *We define the spread σ_{uni} with $\sigma_{uni}(n) = 0$ for every $n \in \mathbb{N}$. σ_{uni} is called the universal spread and is obviously equal to $\mathbb{N}^{\mathbb{N}}$.*

Note that $\mathbb{R}$ is not a spread. But, the set of all regular real numbers is a spread.

Definition 2.1.4. *We define the spread σ_{reg} as follows:*

$$\alpha \in \sigma_{reg} \iff \forall n \in \mathbb{N}\ [\alpha(n) \in \lambda \wedge \alpha(n+1) \sqsubset \alpha(n)]$$

σ_{reg} is called the regular spread.

For the regular spread we have an important lemma.

Lemma 2.1.5. *For the spread σ_{reg} we have the following two properties:*

(i) Every $\alpha \in \sigma_{reg}$ is a real number, and

(ii) for every real number x there exists $\alpha \in \sigma_{reg}$ such that $\alpha \equiv x$.

Proof. First we will show (i). Suppose $\alpha \in \sigma_{\text{reg}}$. Then $\alpha(n) \in \lambda$ for every $n \in \mathbb{N}$. So there exists an $m \in \mathbb{N}$ such that $\alpha(n) \in \lambda_m$, which means $\alpha(n) \in S$. Also for every $n \in \mathbb{N}$ we have $\alpha(n) \sqsubset \alpha(n+1)$ which implies $\alpha(n) \sqsubseteq \alpha(n+1)$. Now pick $m \in \mathbb{N}$, then $\alpha(m) \in \lambda_k$ for a $k \geq m$. This is easily proven by induction. So $l(\alpha(m)) \leq 2^{-k} \leq 2^{-m}$.
Now we will show (ii). Take any real number $x \in \mathbb{R}$. We will define α inductively by making sure that for each $n \geq 1$:

(i) $\alpha(n) \in \lambda$, and

(ii) $\alpha(n) \sqsubset \alpha(n-1)$, and

(iii) there exists $m \in \mathbb{N}$ such that $x(m) \sqsubseteq \alpha(n)$

Obviously, (i) and (ii) ensure $\alpha \in \sigma_{\text{reg}}$. Furthermore we will show (iii) ensures $\alpha \equiv x$.
Now we construct α. First find $m \in \mathbb{N}$ such that $l(x(m)) \leq \frac{1}{4}$. We start with $\alpha(0) = (a, a+1)$, where $2a \in \mathbb{Z}$ and such that $a \leq x'(m)$ but $a + \frac{1}{2} > x'(m)$. Then $a + 1 > x''(m)$. Now suppose $\alpha(0), \alpha(1), \ldots, \alpha(n-1)$ are defined in the previous steps such that (i), (ii) and (iii) hold. Find k such that $\alpha(n-1) \in \lambda_k$. To assure (i) and (ii) for $\alpha(n)$ we would like $\alpha(n) \in \lambda_p$ with $p > k$. Find $m \in \mathbb{N}$ such that $x(m) \sqsubseteq \alpha(n-1)$. Then $x(m+1) \sqsubset \alpha(n-1)$. Decide $x'(m+1) - x'(m) \leq x''(m) - x''(m+1)$ or $x''(m) - x''(m+1) \leq x'(m+1) - x'(m)$.

- Suppose $x'(m+1) - x'(m) \leq x''(m) - x''(m+1)$. This means $x'(m+1) - x'(m) < \frac{1}{2} l(x(m)) < \frac{1}{2^{k-1}}$. Find $p \in \mathbb{N}$ such that $x'(m+1) - x'(m) > \frac{1}{2^{p-1}}$ but $x'(m+1) - x'(m) \leq \frac{1}{2^{p-2}}$. Obviously $p > k$.
  Now find $m' > m+1$ such that $l(x(m')) \leq \frac{1}{2^{p+1}}$. Find $q \in \mathbb{Q}$ such that $q 2^{p+1} \in \mathbb{Z}$, $q \leq x'(m')$, but $q + \frac{1}{2^{p+1}} > x'(m')$. Then $q + \frac{1}{2^p} > x''(m')$ so $x(m') \sqsubseteq (q, q + \frac{1}{2^p})$.
  Furthermore $q > x'(m)$, since suppose $q \leq x'(m)$ then $q + \frac{1}{2^{p+1}} \leq x'(m) + \frac{1}{2^{p+1}} < x'(m+1) < x'(m')$, which is a contradiction. So $q > \alpha'(n-1)$.
  Also $q + \frac{1}{2^p} < x''(m)$ since suppose $q + \frac{1}{2^p} \geq x''(m)$ then $\frac{1}{2^p} \geq l(x(m')) + \big(x''(m) - x''(m+1)\big)$. This is a contradiction since $x''(m) - x''(m+1) \geq x'(m+1) - x'(m) > \frac{1}{2^{p-1}} > \frac{1}{2^p}$.
  Define $\alpha'(n) = q$ and $\alpha''(n) = \alpha'(n) + \frac{1}{2^p}$. Then $\alpha(n) \in \lambda_{p+1}$, $\alpha(n) \sqsubset \alpha(n-1)$, $x(m') \sqsubseteq \alpha(n)$.

- Suppose $x''(m) - x''(m+1) \leq x'(m+1) - x'(m)$. We can do something similar by finding $p \in \mathbb{N}$ such that $x''(m) - x''(m+1) > \frac{1}{2^{p-1}}$ but $x''(m) - x''(m+1) \leq \frac{1}{2^{p-2}}$.

We claim $\alpha \equiv x$. By (iii), for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $x(m) \sqsubseteq \alpha(n)$. Now pick $n \in \mathbb{N}$. Define $l_n = l(x(n))$. Find m such that $l(\alpha(m)) \leq l_n$ and find k such that $x(k) \sqsubseteq \alpha(m)$. Then $l(x(k)) < l_n$ so $x(k) \sqsubset x(n)$. There are three cases:

- $\alpha(m) \sqsubseteq x(n)$, then we are done.

- $x'(n) \leq \alpha'(m) \leq x'(k) < x''(k) < x''(n) \leq \alpha''(m)$. Now, for every $p > m$ we have $\alpha'(p) < x''(k)$. Since, suppose $\alpha'(p) \geq x''(k)$ then there can not exists $b \in \mathbb{N}$ such that $x(b) \sqsubseteq \alpha(p)$. Now find $p > m$ such that $l(\alpha(p)) < x''(n) - x''(k)$. Then $\alpha''(p) < x''(n)$ and since $\alpha'(m) < \alpha'(p)$ and $x'(n) \leq \alpha'(m)$ also $x'(n) < \alpha'(p)$ so $\alpha(p) \sqsubseteq x(n)$.

- $\alpha'(m) \leq x'(n) < x'(k) < x''(k) \leq \alpha''(m) \leq x''(n)$. This is similar as the previous case.

The above procedure defines a function $F_{\text{reg}} : \mathbb{R} \to \sigma_{\text{reg}}$. So for every $x \in \mathbb{R}$ we find $F_{\text{reg}}(x)$ with the above procedure. □

We define one more spread.

Definition 2.1.6. *Fix an enumeration $\{q_0, q_1, q_2, \ldots\}$ of $\mathbb{Q}$. We define the spread σ_{irr} as follows:*

$$\alpha \in \sigma_{irr} \iff \alpha \in \sigma_{reg} \wedge \forall n \in \mathbb{N}[q_n < \alpha'(n) \vee q_n > \alpha''(n)]$$

σ_{irr} is called the spread of the positively irrational numbers.

With the definition of a spread we can introduce the continuity principle, which we will introduce as an axiom. We will need the continuity principle to prove that every real-valued total function $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Axiom 2.1.7. ***(Continuity principle)****. Suppose σ is a spread and $A \subseteq \sigma \times \mathbb{N}$.*
If $\forall \alpha \in \sigma \exists n \in \mathbb{N}[A(\alpha, n)]$ then $\forall \alpha \in \sigma \exists n, m \in \mathbb{N} \forall \beta \in \sigma[A(\alpha, n) \wedge \bar{\alpha}(m) = \bar{\beta}(m) \implies A(\beta, n)]$

We now define what a real-valued total function is and when a real-valued total function is continuous.

Definition 2.1.8. *A **real-valued function** f ($f : \mathbb{R} \to \mathbb{R}$) is a method such that, for every $x \in \mathbb{R}$ we can construct $f(x) \in \mathbb{R}$ and such that for every $x, x' \in \mathbb{R}$, if $x \equiv x'$ then $f(x) \equiv f(x')$.*

When we say $f : \mathbb{R} \to \mathbb{R}$ is a function we mean f is a real-valued total function.

Definition 2.1.9. *Suppose $f : \mathbb{R} \to \mathbb{R}$, then f is **continuous** if $\forall x \in \mathbb{R} \forall n \in \mathbb{N} \exists m \in \mathbb{N} \forall y \in \mathbb{R}\ \left[|x - y| \leq \frac{1}{2^m} \implies |f(x) - f(y)| \leq \frac{1}{2^n}\right]$.*

Definition 2.1.10. *Suppose $f : \mathbb{R} \to \mathbb{R}$, then f is **discontinuous** if $\exists x \in \mathbb{R} \exists n \in \mathbb{N} \forall m \in \mathbb{N} \exists y \in \mathbb{R}\ \left[|x - y| \leq \frac{1}{2^m}\ \wedge\ |f(x) - f(y)| > \frac{1}{2^n}\right]$.*

We will now prove that every function $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Theorem 2.1.11. ***(Continuity theorem)****. Suppose $f : \mathbb{R} \to \mathbb{R}$ is function. Then f is continuous.*

Proof. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. We will define a special function $f' : \sigma_{\text{reg}} \to \sigma_{\text{reg}}$ such that for every $\alpha \in \sigma_{\text{reg}}$, $f(\alpha) \equiv f'(\alpha)$. For every $\alpha \in \sigma_{\text{reg}}$, define $f'(\alpha) = F_{\text{reg}}(f(\alpha))$.
Using f' we will prove that f is continuous. Suppose $x \in \mathbb{R}$ and $m \in \mathbb{N}$. We want to find $n \in \mathbb{N}$ such that for every $y \in \mathbb{R}$ if $|x - y| < \frac{1}{n}$ then $|f(x) - f(y)| < \frac{1}{m}$.
Find $\alpha \in \sigma_{\text{reg}}$ such that $\alpha \equiv x$. Notice that for every $\alpha \in \sigma_{\text{reg}}$ there exists $k \in \mathbb{N}$ such that $f'(\alpha)(m + 1) = k$. Thus, by the continuity principle, we can find a $p \in \mathbb{N}$ such that for every $\beta \in \sigma_{\text{reg}}$ if $\bar{\beta}p = \bar{\alpha}p$ then $f'(\beta)(m + 1) = f'(\alpha)(m + 1)$. We have $\alpha(p) \sqsubset \alpha(p - 1)$. Define $\delta := \min(\alpha'(p) - \alpha'(p - 1), \alpha''(p - 1) - \alpha''(p))$ and find $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. We claim this n is such that for every $y \in \mathbb{R}$ if $|x - y| < \frac{1}{n}$ then $|f(x) - f(y)| < \frac{1}{m}$. Suppose $y \in \mathbb{R}$ and $|x-y| < \frac{1}{n}$. Find $\beta \in \sigma_{\text{reg}}$ such that $\beta \equiv y$ and $\bar{\beta}p = \bar{\alpha}p$. This gives $f'(\beta)(m+1) = f'(\alpha)(m+1)$. Since $f'(\alpha) \in \sigma)\text{reg}$ we have $l(f'(\alpha)(m + 1) \leq 2^{-m-1}$, which is easily shown with induction. Also $f'(\alpha)'(m + 1) \leq f'(\alpha) \leq f'(\alpha)''(m + 1)$ and $f'(\alpha)'(m + 1) \leq f'(\beta) \leq f'(\alpha)''(m + 1)$ and $f'(\alpha) \equiv f(x)$ and $f'(\beta) \equiv f(y)$ so $|f(x) - f(y)| \leq \frac{1}{m}$. □

We now prove three more lemmas related to continuous functions which will become useful during this thesis.

Lemma 2.1.12. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and suppose $x, y \in \mathbb{R}$. If $f(x) \# f(y)$ then $x \# y$.*

Proof. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and suppose $x, y \in \mathbb{R}$ and $f(x) \# f(y)$. Then, by lemma 1.2.5, there exists an $m \in \mathbb{N}$ such that $|f(x) - f(y)| > \frac{1}{m}$. By the continuity of f we know, at x, $\forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall y \in \mathbb{R}$ $[\ |x - y| \leq \frac{1}{n} \to |f(x) - f(y)| \leq \frac{1}{k}]$. Find $n \in \mathbb{N}$ such that for all $y \in \mathbb{R}$ if $|x - y| \leq \frac{1}{n}$ then $|f(x) - f(y)| \leq \frac{1}{m}$. This gives us $|x - y| > \frac{1}{n}$ and thus $x \# y$. □

The following lemma is an adapted version of the classical intermediate value theorem.

Lemma 2.1.13. ***(Approximate intermediate value theorem)*** *Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) = d$, $f(b) = e$ and $|a - b| \leq 1$. If $d < e$ then for every $c \in [d, e]$ and every $n \in \mathbb{N}$ there exists $c_n \in [a, b]$ such that $|f(c_n) - c| \leq \frac{1}{2^n}$. If $e < d$ then for every $c \in [e, d]$ and every $n \in \mathbb{N}$ there exists $c_n \in [a, b]$ such that $|f(c_n) - c| \leq \frac{1}{2^n}$.*

Proof. Suppose $d < e$. Pick $c \in [d, e]$ and $n \in \mathbb{N}$. We will define sequences $a_1, a_2, \ldots, b_1, b_2, \ldots$ and $c_1, c_2, \ldots$. First define $a_1 := a$, $b_1 := b$ and $c_1 := \frac{a+b}{2}$. Now suppose $k \in \mathbb{N}$ and a_k, b_k and c_k are defined. Since $c - \frac{1}{2^{n+2}} < c + \frac{1}{2^{n+2}}$ we know, either $f(c_k) > c - \frac{1}{2^{n+2}}$ or $f(c_k) < c + \frac{1}{2^{n+2}}$. Suppose $f(c_k) > c - \frac{1}{2^{n+2}}$ then define $a_{k+1} := a_k, b_{k+1} := c_k$ and $c_{k+1} := \frac{a_{k+1}+b_{k+1}}{2}$. Suppose a_{k+1}, b_{k+1} and c_{k+1} have not been defined yet, then $f(c_k) < c + \frac{1}{2^{n+2}}$. Define $a_{k+1} := c_k, b_{k+1} := b_k$ and $c_{k+1} := \frac{a_{k+1}+b_{k+1}}{2}$. We now have the following:

(i) For every $k \in \mathbb{N}$, $f(a_k) < c + \frac{1}{2^{n+2}}$

(ii) For every $k \in \mathbb{N}$, $f(b_k) > c - \frac{1}{2^{n+2}}$

(iii) For every $k \in \mathbb{N}$, $c_k - a_k \leq \frac{1}{2^k}$ and $b_k - c_k \leq \frac{1}{2^k}$

We will prove (i), (ii) and (iii).

(i) $f(a_1) = f(a) = d < c + \frac{1}{2^{n+2}}$. Now suppose $k \in \mathbb{N}$ and suppose we have proven $f(a_l) < c + \frac{1}{2^{n+2}}$ for every $l \leq k$. Suppose $f(c_k) > c - \frac{1}{2^{n+2}}$, then $a_{k+1} = a_k$ so $f(a_{k+1}) = f(a_k) < c + \frac{1}{2^{n+2}}$, by the induction hypothesis. Now suppose $f(c_k) < c + \frac{1}{2^{n+2}}$, then $a_{k+1} = c_k$ so $f(a_{k+1}) = f(c_k) < c + \frac{1}{2^{n+2}}$.

(ii) $f(b_1) = f(b) = e > c - \frac{1}{2^{n+2}}$. Now suppose $k < m$ and suppose we have proven $f(b_l) > c - \frac{1}{2^{n+2}}$ for every $l \leq k$. Suppose $f(c_k) > c - \frac{1}{2^{n+2}}$, then $b_{k+1} = c_k$ so $f(b_{k+1}) = f(c_k) > c - \frac{1}{2^{n+2}}$. Now suppose $f(c_k) < c + \frac{1}{2^{n+2}}$, then $b_{k+1} = b_k$ so $f(b_{k+1}) = f(b_k) > c - \frac{1}{2^{n+2}}$, by the induction hypothesis.

(iii) This is easily shown with induction.

We define a real number x such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = x$. First define $x(0) = (a_0 - 1, b_0 + 1)$. Now suppose $x(0), x(1), \ldots, x(n-1)$ are defined. Suppose $a_n = a_{n-1}$, then define $x'(n) = x'(n-1) + \frac{1}{2}l(x'(n-1), a_{n-1})$ and $x''(n) = b_{n-1}$. Suppose $a_n > a_{n-1}$, then define $x'(n) = a_{n-1}$ and $x''(n) = x''(n-1) - \frac{1}{2}l(b_{n-1}, x''(n-1))$. It is easily shown with induction that $a_n - x'(n) = \frac{1}{2^n}$ and $x''(n) - b_n = \frac{1}{2^n}$. We now show $x \in \mathbb{R}$. Obviously, for every $n \in \mathbb{N}$, $x(n) \in S$ and $x(n) \sqsubseteq x(n+1)$. Pick $m \in \mathbb{N}$. Find $k \in \mathbb{N}$ such that $\frac{3}{2^k} \leq \frac{1}{2^m}$ and find $n \geq k$ such that $b_n - a_n \leq \frac{1}{2^k}$. Then $l(x(n)) = (x''(n) - b_n) + (b_n - a_n) + (a_n - x'(n)) \leq \frac{3}{2^k} \leq \frac{1}{2^m}$. Now, for every $n \in \mathbb{N}$, $k \geq n$ $x'(n) < x < x''(n)$ and $x'(n) < a_k < b_k < x'(n)$. So, for every $n \in \mathbb{N}$, $k \geq n$, $|x - a_k| \leq l(x(n))$. Now, pick $m \in mathbbN$ and find $l \in \mathbb{N}$ such that $l(x(l)) \leq \frac{1}{2^m}$. Then, for every $k \geq l$, $|x - a_k| \leq \frac{1}{2^m}$. So, for every $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for every $k \geq l$, $|x - a_k| \leq \frac{1}{2^m}$. With a similar argument, we prove for every $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for every $k \geq l$, $|x - b_k| \leq \frac{1}{2^m}$ and for every $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for every $k \geq l$, $|x - c_k| \leq \frac{1}{2^m}$.
Find $m \in \mathbb{N}$ such that for all $y \in [a, b]$ if $|x - y| \leq \frac{1}{2^m}$ then $|f(x) - f(y)| \leq \frac{1}{2^{n+2}}$. Find $n \in \mathbb{N}$, $k \geq n$ such that $|x - a_k| \leq \frac{1}{2^m}$ and $l \in \mathbb{N}$, $p \geq l$ such that $|x - b_p| \leq \frac{1}{2^m}$ and $t \in \mathbb{N}$, $q \geq t$ such that $|x - c_q| \leq \frac{1}{2^m}$. Then $|f(x) - f(a_k)| \leq \frac{1}{2^{n+2}}$, $|f(x) - f(b_p| \leq \frac{1}{2^{n+2}}$ and $|f(x) - f(c_q)| \leq \frac{1}{2^{n+2}}$. This means $f(a_k) - \frac{1}{2^{n+2}} \leq f(x) \leq f(a_k) + \frac{1}{2^{n+2}}$ and $f(b_p) - \frac{1}{2^{n+2}} \leq f(x) \leq f(b_p) + \frac{1}{2^{n+2}}$. Combining this with (i) and (ii) we get $c - \frac{1}{2^{n+1}} < f(b_p) - \frac{1}{2^{n+2}} < f(x) < f(a_k) + \frac{1}{2^{n+2}} < c + \frac{1}{2^{n+1}}$ and thus $|f(x) - c| \leq \frac{1}{2^{n+1}}$. So, with lemma 1.1.7, we have $|f(c_q) - c| \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} \leq \frac{1}{2^n}$.
The case where $e < d$ is similar. $\square$

An important consequence of the continuity theorem is that the continuum is indecomposable (*Brouwer: unzerlegbar*).

Lemma 2.1.14. *Suppose $A, B \subseteq \mathbb{R}$ are such that $\mathbb{R} = A \cup B$ and $A \cap B = \emptyset$. Then $\mathbb{R} = A$ or $\mathbb{R} = B$.*

Proof. Suppose $A, B \subseteq \mathbb{R}$ are such that $\mathbb{R} = A \cup B$ and $A \cap B = \emptyset$. This means, for every $x \in \mathbb{R}[x \in A \ \vee \ x \in B]$. Define $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$. Since for every $x \in \mathbb{R}[x \in A \ \vee \ x \in B]$ and for every $x \in \mathbb{R}\neg[x \in A \ \wedge \ x \in B]$ this is a well defined total function. So, by the continuity theorem f is continuous. Now pick $x_1 \in \mathbb{R}$ and decide $x_1 \in A$ or $x_1 \in b$ Suppose, without loss of generality, $x_1 \in A$. We will then prove $\mathbb{R} = A$. Suppose we find $x_2 \in B$ such that $x_1 \# x_2$. Also suppose, without loss of generality, $x_1 < x_2$. We now have $f[x_1, x_2] \to \mathbb{R}$ continuous and $f(x_1) = 0$ and $f(x_2) = 1$. By the approximate intermediate value theorem, there exists $x \in [x_1, x_2]$ such that $|f(x) - \frac{1}{2}| \leq \frac{1}{4}$. But for every $x \in [x_1, x_2]$, $f(x) = 0$ or $f(x) = 1$, so this is a contradiction. So $\neg x_2 \in B$ which means $x_2 \in A$. Thus for every $x \in \mathbb{R}$, $x \in A$, so $A = \mathbb{R}$. □

An interesting consequence of this lemma is that we can prove $\neg\forall x \in \mathbb{R}[x \in \mathbb{Q} \vee x \notin \mathbb{Q}]$. See section 2 of [5].

2.2 Fans

In this section we will introduce fans, the fan theorem and the extended fan theorem. Also, we will state that every real-valued total function on $[0, 1]$ is uniformly continuous.
First we will define what a fan is.

Definition 2.2.1. *A **fan-law** is a function $\tau : \mathbb{N} \to \{0, 1\}$ such that:*

(i) τ is a spread-law, and

(ii) $\forall s \in \mathbb{N}\big[\tau(s) = 0 \implies \exists m \in \mathbb{N}\forall n \in \mathbb{N}[\tau(s* < n >) = 0 \implies n < m]\big]$

Definition 2.2.2. *Suppose $\beta \in \mathbb{N}^{\mathbb{N}}$. Define $\beta \in \tau$ iff for every $n \in \mathbb{N}$, $\tau(\bar{\beta}n) = 0$. The set $\{\beta \in \mathbb{N}^{\mathbb{N}} | \beta \in \tau\}$ is called a **fan**. We will also use τ to refer to this set.*

Every fan τ is thus a spread for which there are only finitely many choices for each next step in the creation of any $\beta \in \tau$. This means the nodes in the tree defined by the fan-law have finitely many children.
Note that every closed interval of $\mathbb{R}$ coincides with a fan.
There are two more important theorems, namely the Fan theorem and the Extended fan theorem. Brouwer used the extended Fan theorem to prove that every function on a closed interval is uniformly continuous. To give the Fan theorem we need one more definition.

Definition 2.2.3. *Suppose $B \subseteq \mathbb{N}$ and $X \subseteq \mathbb{N}^{\mathbb{N}}$. B is a **bar** in X if and only if for every $\alpha \in X$ there exists $n \in \mathbb{N}$ such that $\bar{\alpha}n \in B$.*

Theorem 2.2.4. ***(Fan theorem)**. Suppose τ is a fan and $B \subseteq \mathbb{N}$ is a bar in τ then there exists a finite $C \subseteq B$ which is a bar in τ.*

Using the Fan theorem and the continuity principle we can prove the Extended fan theorem.

Theorem 2.2.5. ***(Extended fan theorem)**. Suppose τ is a fan and $A \subset \tau \times \mathbb{N}$ such that $\forall\beta \in \tau\exists n \in \mathbb{N}[(\beta, n) \in A]$. Then we can find $M \in \mathbb{N}$ such that $\forall\beta \in \tau\exists n \in \mathbb{N}[n \leq M \wedge (\beta, n) \in A]$.*

With the Extended fan theorem we can prove that every real-valued total function on $[0,1]$ is uniformly continuous. When we say $f : [0,1] \to \mathbb{R}$ we mean f is a real-valued total function on $[0,1]$. First we will define what it means for a function to be uniformly continuous.

Definition 2.2.6. *Suppose $f : \mathbb{R} \to \mathbb{R}$, then f is* ***uniformly continuous*** *if* $\forall n \in \mathbb{N} \exists m \in \mathbb{N} \forall x, y \in \mathbb{R} \left[|x - y| \leq \frac{1}{2^m} \implies |f(x) - f(y)| \leq \frac{1}{2^n} \right]$.

Theorem 2.2.7. ***(Uniform Continuity Theorem)****. Suppose $f : [0,1] \to \mathbb{R}$ is a function. Then f is uniformly continuous.*

The indecomposability of $\mathbb{R}$ is also true for $[0,1]$.

Lemma 2.2.8. *Suppose $A, B \subseteq [0,1]$ are such that $[0,1] = A \cup B$ and $A \cap B = \emptyset$, then $[0,1] = A$ or $[0,1] = B$.*

Proof. This will be similar to the proof of lemma 2.1.14. □

We will now prove one more lemma related to uniformly continuous functions. It proves that every uniformly continuous function is monotone. This lemma will become useful in this thesis. For this we need one more definition.

Definition 2.2.9. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. We call f an injection if for every $x, y \in \mathbb{R}$ if $x \# y$ then $f(x) \# f(y)$.*

Lemma 2.2.10. *Suppose $f : [0,1] \to [0,1]$ is a continuous injection such that f^{-1} is also continuous. Then, for all $x, y \in [0,1]$ if $x < y$ then $f(x) < f(y)$ or for all $x, y \in [0,1]$ if $x < y$ then $f(x) > f(y)$.*

Proof. Suppose $f : [0,1] \to [0,1]$ is a continuous injection and suppose f^{-1} is also continuous. Consider 0 and 1. Since f is an injection and since $0 \# 1$ we know $f(0) \# f(1)$. This means $f(0) < f(1)$ or $f(1) < f(0)$. If $f(0) < f(1)$, then for all $x, y \in [0,1]$ if $x < y$ then $f(x) < f(y)$. And if $f(1) < f(0)$ then for all $x, y \in [0,1]$ if $x < y$ then $f(y) < f(x)$. Now suppose $f(0) < f(1)$. The other case is similar.
First we will prove, for every $x \in [0,1]$ if $0 < x$ then $f(0) < f(x)$. Since $0 < x$ we have $f(0) \# f(x)$, so $f(0) < f(x)$ or $f(x) < f(0)$. We will prove $\neg(f(x) < f(0))$ so $f(0) < f(x)$. For this, suppose $f(x) < f(0)$. This means $f(x) < f(0) < f(y)$. Now look at $f \restriction [x, y] : [x, y] \to \mathbb{R}$. By the approximate intermediate value theorem, for every $n \in \mathbb{N}$ there exists c_n such that $|f(c_n) - f(0)| \leq \frac{1}{n}$. Also, since f^{-1} is continuous, for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that if $|a - b| \leq \frac{1}{n}$ then $|f(a) - f(b)| \leq \frac{1}{m}$. Thus, for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $|c_n - 0| \leq \frac{1}{m}$. This means $\lim_{n \to \infty} c_n = 0$, so $0 \in [x, y]$, which is a contradiction.
Similarly we can show, for every $y \in [0,1]$ if $y < 1$ then $f(y) < f(1)$.
Now we will prove, for every $x, y \in [0,1]$ if $x < y$ then $f(x) < f(y)$. Pick $x, y \in [0,1]$ such that $x < y$. Then, since f is an injection and $x \# y$, $f(x) \# f(y)$. This means either $f(x) < f(y)$ or $f(y) < f(x)$. Also, we can prove $\neg(f(y) < f(x))$ so we must have $f(x) < f(y)$. We will now prove $\neg(f(y) < f(x))$. Suppose $f(y) < f(x)$. Then consider $f \restriction [0, x] : [0, x] \to \mathbb{R}$. Since $f(0) \leq f(y) < f(x)$, by the intermediate value theorem, for every $n \in \mathbb{N}$ there exists c_n such that $|f(c_n) - f(y)| < \frac{1}{n}$. Again, for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that if $|a - b| \leq \frac{1}{2^n}$ then $|f^{-1}(a) - f^{-1}(b)| \leq \frac{1}{2^m}$. Thus, for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $|c_n - y| \leq \frac{1}{2^m}$. This means $\lim_{n \to \infty} c_n = y$, thus $y \in [0, x]$, which is a contradiction. □

3 Measure theory

In this chapter we will introduce intuitionistic measure theory. In the first section we will define some notions needed to define almost full sets and measurable functions and sets. In the second section we will define almost full sets and in the third section we will define measurable functions and sets.

3.1 Intervals, rectangles and regions

We will need to define a couple of things before we are able to discuss almost full sets, measurable functions and measurable sets. Let length : $\mathbb{N} \to \mathbb{N}$ be a function, where length(n) is the length of the sequence of natural numbers corresponding to n via the bijection $< >$ introduced in the introduction. Also, for every $n \in \mathbb{N}$ and every $i \leq \text{length}(n)$: n_{i-1} is the i-th number of the sequence of natural numbers corresponding to n via $< >$.

Definition 3.1.1. *Suppose $v \in S$, where S is the set of (code numbers of) rational segments introduced in chapter 1. Suppose $x \in \mathbb{R}$.*

(i) $x \; \varepsilon_0 \; v$ if and only if $\forall n \in \mathbb{N} \; [x'(n) \leq v'' \wedge v' \leq x''(n)]$

(ii) $x \; \varepsilon_1 \; v$ if and only if $\exists n \in \mathbb{N} \; [v' < x'(n) \leq x''(n) < v'']$

*Then $\{x \in \mathbb{R} \mid x \; \varepsilon_0 \; v\}$ is the **closed interval** defined by v and $\{x \in \mathbb{R} \mid x \; \varepsilon_1 \; v\}$ is the **open interval** defined by v.*

We also define an intersection of two intervals in S.

Definition 3.1.2. *For $s, t \in S$ we define s and t **touch** if and only if $\max(s', t') \leq \min(s'', t'')$ and s and t **miss** if and only if $\max(s', t') > \min(s'', t'')$.*
Also, we define:

$$s \cap t = \begin{cases} (\max(s', t'), \min(s'', t'')) & \text{if } s \text{ and } t \text{ touch} \\ \perp & \text{else} \end{cases}$$

We now define partial functions, since the definition of a measurable function is for partial functions.

Definition 3.1.3. *$f : [0,1] \to \mathbb{R}$ is a **partial function** if there exists $X \subseteq [0,1]$ such that for every $x \in X$ we can construct $f(x) \in \mathbb{R}$ and such that for every $x, x' \in X$, if $x \equiv x'$ then $f(x) \equiv f(x')$.*
We define $\text{dom}(f) := \{x \in [0,1] \mid \exists y \in \mathbb{R} \; [(x,y) \in f]\}$ to be the domain of f.

So $f : [0,1] \to \mathbb{R}$ is a partial function if it is function on a subset of $[0,1]$.

Definition 3.1.4. *A **rational rectangle** is a natural number $v \in \mathbb{N}$ such that $\text{length}(v) \geq 2$ and such that $v_0 \in S$ and $v_1 \in S$. We define $R = \{v \in \mathbb{N} \mid v \text{ is a rational rectangle}\}$.*

Definition 3.1.5. *Suppose $v \in R$. We define $\text{Ar}(v) := (v_0'' - v_0')(v_1'' - v_1')$ to be the area of v.*

Definition 3.1.6. *An **elementary set of rectangles** is a natural number $v \in \mathbb{N}$ such that:*

(i) for every $i < \text{length}(v)$ $[v_i \in R]$, and

(ii) the sequence $(v_0)_0, \ldots, (v_{\text{length}(v)-1})_0$ is a partition of $[0,1]$, that is:
$(v_0)_0' = 0, (v_0)_0'' = (v_1)_0', (v_1)_0'' = (v_2)_0', \ldots, (v_{\text{length}(v)-1})_0'' = 1$.

We now define when an elementary set of rectangles captures a partial function. For an intuitive notion see figure 4.

Definition 3.1.7. *Suppose $f : [0,1] \to \mathbb{R}$ is a partial function and v is an elementary set of rectangles. We say v* ***captures*** *f if and only if for all $x \in \mathrm{dom}(f)$ and for all $i < \mathrm{length}(v)$ if $x \; \varepsilon_0 \; (v_i)_0$ then $f(x) \; \varepsilon_0 \; (v_i)_1$.*

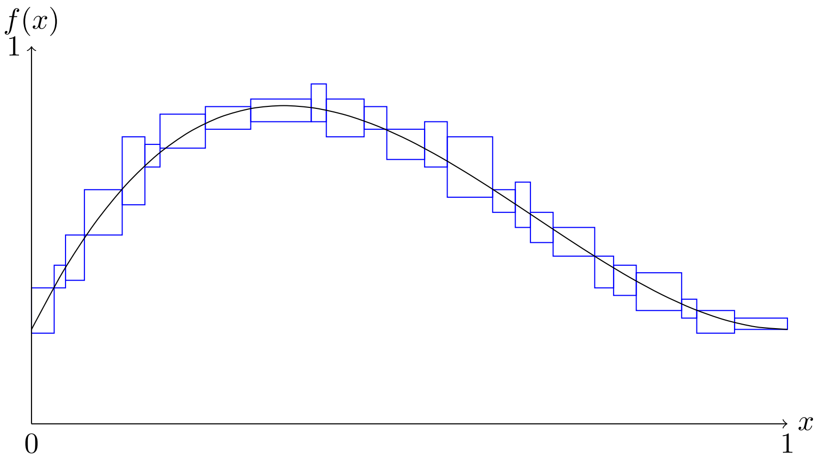

Figure 4: The set of elementary rectangles capture f.

Definition 3.1.8. *Suppose v is an elementary set of rectangles. We define:*

(i) The area of v as $\mathrm{Ar}^*(v) = \sum_{i=0}^{\mathrm{length}(v)-1} \mathrm{Ar}(v_i)$.

(ii) $\underline{I}(v) = \sum_{i=0}^{\mathrm{length}(v)-1} \big((v_i)_0'' - (v_i)_0'\big)(v_i)_1'$

(iii) $\overline{I}(v) = \sum_{i=0}^{\mathrm{length}(v)-1} \big((v_i)_0'' - (v_i)_0'\big)(v_i)_1''$

Definition 3.1.9. *Suppose $a, m \in \mathbb{N}$ such that $\mathrm{length}(a) = m$ and $\forall i < m \; [a_i \in S]$. We define $\mu(a)$ to be the total length of the rational segments $a_0, a_1, \ldots, a_{m-1}$ where 'double covered intervals are not counted twice'. That is, if $a_0, a_1, \ldots, a_{m-1}$ do not intersect pairwise then $\mu(a) = l(a_0) + l(a_1) + \cdots + l(a_{m-1})$, else:*

$$\mu(a) = \sum_{k=0}^{m-1} \mu\big(a_k \setminus \bigcup_{i=0}^{k-1} a_i\big)$$

We now define the intuitionistic notion of an open set. We call this a region.

Definition 3.1.10. *Suppose $\alpha = \alpha(0), \alpha(1), \alpha(2), \ldots$ is an infinite sequence of code numbers of rational segments, that is, for every $n \in \mathbb{N} : \alpha(n) \in S$. We call $\mathcal{R}(\alpha) := \{x \in \mathbb{R} \mid \exists m \in \mathbb{N} \; [x \; \varepsilon_1 \; \alpha(m)]\}$ the* ***region*** *defined by α.*

Definition 3.1.11. *We define X is a* ***measurable region*** *if and only if there exists an infinite sequence $\alpha = \alpha(0), \alpha(1), \alpha(2), \ldots$ of code numbers of rational segments such that $X = \mathcal{R}(\alpha)$ and such that the sequence $\mu(\bar{\alpha}1), \mu(\bar{\alpha}2), \ldots$ converges. If X is a measurable region we call $\mu(X) := \lim_{n\to\infty} \mu(\bar{\alpha}n)$ the measure of X.*

The following lemma shows that this definition makes sense.

Lemma 3.1.12. *Suppose $\alpha = \alpha(0), \alpha(1), \alpha(2), \ldots$ is an infinite sequences of code numbers of rational segments such that $\mu(\bar{\alpha}1), \mu(\bar{\alpha}2), \ldots$ converges. Suppose also, $\beta = \beta(0), \beta(1), \beta(2), \ldots$ is an infinite sequence of code numbers of rational segments and $\mathcal{R}(\alpha) = \mathcal{R}(\beta)$. Then $\mu(\bar{\beta}1), \mu(\bar{\beta}2), \ldots$ converges and $\lim_{n\to\infty} \mu(\bar{\alpha}n) = \lim_{n\to\infty} \mu(\bar{\beta}n)$*

Proof. We know α is such that $\mu(\bar{\alpha}1), \mu(\bar{\alpha}2), \ldots$ converges to $\mu(\mathcal{R}(\alpha))$. We want to show: $\mu(\bar{\beta}1), \mu(\bar{\beta}2), \ldots$ converges to $\mu(\mathcal{R}(\alpha))$. It suffices to prove:

(i) For every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $\mu(\bar{\beta}k) > \mu(\mathcal{R}(\alpha)) - \frac{1}{2^n}$, and

(ii) for every $m, l \in \mathbb{N}$, $\mu(\bar{\beta}l) - \frac{1}{2^m} \leq \mu(\mathcal{R}(\alpha))$.

We will now prove (i) and (ii).

(i) We first prove, for every $l, m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $\mu(\bar{\beta}k) > \mu(\bar{\alpha}l) - \frac{1}{2^m}$. Pick $l, m \in \mathbb{N}$ and find $n \in \mathbb{N}$ such that $\frac{2l}{2^n} < \frac{1}{2^m}$. Consider $\bigcup_{i<l}[\alpha(i)' + \frac{1}{2^n}, \alpha(i)'' - \frac{1}{2^n}]$. Since $\mathcal{R}(\alpha) = \mathcal{R}(\beta)$ we know, for every $x \in \bigcup_{i<l}[\alpha(i)' + \frac{1}{2^n}, \alpha(i)'' - \frac{1}{2^n}]$ there exists $p \in \mathbb{N}$ such that $x \in \beta(p)$. Since $\bigcup_{i<l}[\alpha(i)' + \frac{1}{2^n}, \alpha(i)'' - \frac{1}{2^n}]$ is a finite union of closed intervals, by the fan theorem, there exists $k \in \mathbb{N}$ such that for every $x \in \bigcup_{i<l}[\alpha(i)' + \frac{1}{2^n}, \alpha(i)'' - \frac{1}{2^n}]$ there exists $p \leq k$ such that $x \in \beta(p)$. This means $\bigcup_{i<l}[\alpha(i)' + \frac{1}{2^n}, \alpha(i)'' - \frac{1}{2^n}] \subseteq \mathcal{R}(\bar{\beta}k)$ and thus $\mu(\bar{\alpha}l) - \frac{1}{2^m} < \mu(\bar{\alpha}l) - \frac{2l}{2^n} <\leq \mu(\bigcup_{i<l}[\alpha(i)' + \frac{1}{2^n}, \alpha(i)'' - \frac{1}{2^n}]) \leq \mu(\bar{\beta}k)$. Thus there exists $k \in \mathbb{N}$ such that $\mu(\bar{\alpha}l) - \frac{1}{2^m} < \mu(\bar{\beta}k)$.
Since $\mu(\bar{\alpha}1), \mu(\bar{\alpha}2), \ldots$ converges to $\mu(\mathcal{R}(\alpha))$, for every $n \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that $\mu(\bar{\alpha}l) > \mu(\mathcal{R}(\alpha)) - \frac{1}{2^n}$. Note: for every $n \in \mathbb{N}$ there exists $l, m \in \mathbb{N}$ such that $\mu(\bar{\alpha}l) - \frac{1}{2^m} > \mu(\mathcal{R}(\alpha) - \frac{1}{2^n}$. Now, pick $n \in \mathbb{N}$ and find $l, m \in \mathbb{N}$ such that $\mu(\bar{\alpha}l) - \frac{1}{2^m} > \mu(\mathcal{R}(A) - \frac{1}{2^n}$. Now find $k \in \mathbb{N}$ such that $\mu(\bar{\beta}k) > \mu(\bar{\alpha}l) - \frac{1}{2^m}$, then $\mu(\bar{\beta}k) > \mu(\mathcal{R}(A) - \frac{1}{2^n}$.

(ii) Suppose m and l are given. We have $\mu\big(\bigcup_{i<l}[\beta'(i) + \frac{1}{l2^m}, \beta''(i) - \frac{1}{l2^m}]\big) \geq \mu(\bar{\beta}l) - \frac{1}{2^m}$. Also, obviously $\bigcup_{i<l}[\beta'(i) + \frac{1}{l2^m}, \beta''(i) - \frac{1}{l2^m}] \subset \mathcal{R}(\beta) = \mathcal{R}(\alpha)$ so $\mu\big(\bigcup_{i<l}[\beta'(i) + \frac{1}{l2^m}, \beta''(i) - \frac{1}{l2^m}]\big) \leq \mu(\mathcal{R}(\alpha))$. This means $\mu(\bar{\beta}l) - \frac{1}{2^m} \leq \mu(\mathcal{R}(\alpha))$

□

Lemma 3.1.13. *Suppose X and Y are measurable regions and $v \in S$. Then:*

(i) $X \cap v$ is a measurable region

(ii) $X \cup Y$ is a measurable region

Proof.

(i) X is a measurable region, so there exists an infinite sequence $\alpha = \alpha(0), \alpha(1), \alpha(2), \ldots$ of code numbers of rational segments such that $\mu(\bar{\alpha}1), \mu(\bar{\alpha}2), \ldots$ converges and such that $\mathcal{R}(\alpha) = X$.
Define, for all $i \in \mathbb{N}$, $\beta(i) = \alpha(i) \cap v$ and $\beta = \beta(0), \beta(1), \beta(2), \ldots$. Then $\mathcal{R}(\beta) = X \cap v$. We will show that $\mu(\bar{\beta}1), \mu(\bar{\beta}2), \ldots$ converges and thus that $X \cap v$ is a measurable region.
Since $\mu(\bar{\alpha}1), \mu(\bar{\alpha}2), \ldots$ converges we know:
$\forall n \in \mathbb{N} \exists m \in \mathbb{N} \, [\sum_{k=m}^{\infty} \mu\Big(\alpha(k) \setminus \bigcup_{i=0}^{k-1} \alpha(i)\Big) \leq \frac{1}{n}]$, thus
$\forall n \in \mathbb{N} \exists m \in \mathbb{N} \, [\sum_{k=m}^{\infty} \mu\Big((\alpha(k) \setminus \bigcup_{i=0}^{k-1} \alpha(i)) \cap v\Big) \leq \frac{1}{n}]$, so
$\forall n \in \mathbb{N} \exists m \in \mathbb{N} \, [\sum_{k=m}^{\infty} \mu\Big((\alpha(k) \cap v) \setminus (\bigcup_{i=0}^{k-1} \alpha(i)) \cap v\Big) \leq \frac{1}{n}]$, so
$\forall n \in \mathbb{N} \exists m \in \mathbb{N} \, [\sum_{k=m}^{\infty} \mu\Big((\alpha(k) \cap v) \setminus \bigcup_{i=0}^{k-1} (\alpha(i) \cap v)\Big) \leq \frac{1}{n}]$.

(ii) X is a measurable region, so there exists an infinite sequence $\alpha = \alpha(0), \alpha(1), \alpha(2), \ldots$ of code numbers of rational segments such that $\mu(\bar{\alpha}1), \mu(\bar{\alpha}2), \ldots$ converges and such that $\mathcal{R}(\alpha) = X$. Y is a measurable region, so there exists an infinite sequence $\beta = \beta(0), \beta(1), \beta(2), \ldots$ of code numbers of rational segments such that $\mu(\bar{\beta}1), \mu(\bar{\beta}2), \ldots$ converges and such that $\mathcal{R}(\beta) = Y$.
Define, for all $i \in \mathbb{N}$, $\gamma(2i) = \alpha(i)$ and $\gamma(2i+1) = \beta(i)$ and $\gamma = \gamma(0), \gamma(1), \gamma(2), \ldots$. Then $\mathcal{R}(\gamma) = X \cup Y$. We will show that $\mu(\bar{\gamma}1), \mu(\bar{\gamma}2), \ldots$ converges and thus that $X \cup Y$ is a measurable region.
$\mu(\bar{\alpha}1), \mu(\bar{\alpha}2), \ldots$ converges, so $\forall n \in \mathbb{N} \exists m \in \mathbb{N} \, [\sum_{j=m}^{\infty} \mu\Big(\alpha(j) \setminus \bigcup_{i=0}^{j-1} \alpha(i)\Big) \leq \frac{1}{2n}]$.
$\mu(\bar{\beta}1), \mu(\bar{\beta}2), \ldots$ converges, so $\forall n \in \mathbb{N} \exists k \in \mathbb{N} \, [\sum_{j=k}^{\infty} \mu\Big(\beta(j) \setminus \bigcup_{i=0}^{j-1} \beta(i)\Big) \leq \frac{1}{2n}]$.
Now pick $n \in \mathbb{N}$ and find $m, k \in \mathbb{N}$ such that $[\sum_{j=m}^{\infty} \mu\Big(\alpha(j) \setminus \bigcup_{i=0}^{j-1} \alpha(i)\Big) \leq \frac{1}{2n}]$ and $[\sum_{j=k}^{\infty} \mu\Big(\beta(j) \setminus \bigcup_{i=0}^{j-1} \beta(i)\Big) \leq \frac{1}{2n}]$. Then:
$[\sum_{j=\max\{m,k\}}^{\infty} \mu\Big(\alpha(j) \setminus \bigcup_{i=0}^{j-1} \alpha(i)\Big) + \sum_{j=\max\{m,k\}}^{\infty} \mu\Big(\beta(j) \setminus \bigcup_{i=0}^{j-1} \beta(i)\Big) \leq \frac{1}{n}]$,so
$[\sum_{j=\max\{m,k\}}^{\infty} \mu\Big(\alpha(j) \setminus \bigcup_{i=0}^{j-1} \alpha(i)\Big) + \mu\Big(\beta(j) \setminus \bigcup_{i=0}^{j-1} \beta(i)\Big) \leq \frac{1}{n}]$, so
$[\sum_{j=\max\{m,k\}}^{\infty} \mu\Big(\alpha(j) \setminus \bigcup_{i=0}^{j-1} \alpha(i) \cup \bigcup_{i=0}^{j-1} \beta(i)\Big) + \mu\Big(\beta(j) \setminus \bigcup_{i=0}^{j-1} \beta(i) \cup \bigcup_{i=0}^{j} \alpha(i)\Big) \leq \frac{1}{n}]$.

□

3.2 Almost full sets

Definition 3.2.1. *Let $X \subseteq [0,1]$. The set X is **almost full** if and only if there exists a sequence $X_0, X_1, X_2, \ldots$ of measurable regions such that:*

(i) for every $n \in \mathbb{N}$, $\mu(X_n) < \frac{1}{2^n}$, and

(ii) for every $n \in \mathbb{N}$, every $x \in [0,1]$ if $x \notin X_n$ then $x \in X$.

The following lemma proves that for every measurable region of measure smaller then 1 we can find an element in the complement. With that we can show that for every set which is almost full we can find an element in the set.

Lemma 3.2.2. *Suppose X is a measurable region and $\mu(X) < 1$, then there exists an $x \in [0,1]$ such that $x \notin X$.*

Proof. X is a measurable region, so there exists an infinite sequence $\alpha = \alpha(0), \alpha(1), \alpha(2), \ldots$ of code numbers of rational segments such that $\mu(\bar{\alpha}1), \mu(\bar{\alpha}2), \ldots$ converges and such that $\mathcal{R}(\alpha) = X$. We will define a real number x as a sequence $x(0), x(1), x(2), \ldots$ of (code numbers of) rational segments such that $x \notin \mathcal{R}(\alpha)$. We will define x such that it has the following properties:

(i) $\forall n \in \mathbb{N}[0 \leq x'(n) < x'(n+1) < x''(n+1) < x''(n) \leq 1]$, and

(ii) $\forall n \in \mathbb{N}[l(x(n+1)) = \frac{1}{2}l(x(n))]$, and

(iii) $\forall n \in \mathbb{N}[\frac{\mu(\mathcal{R}(x(n) \cap \alpha)}{\mu(x(n))} \leq 1 - \frac{1}{2^{n+1}}]$.

Ensuring (i) and (ii) we know that x is a real number and that $x \in [0,1]$. We will show that (iii) ensures $x \notin \mathcal{R}(\alpha)$. Suppose $x \in \mathcal{R}(\alpha)$. Find $m, n \in \mathbb{N}$ such that $\alpha'(n) < x'(m) < x''(m) < \alpha''(n)$. This means $x(m) \sqsubset \alpha(n)$, so $\mu(x(m) \cap \alpha(n)) = \mu(\mathcal{R}(x(m) \cap \alpha)) = \mu(x(m))$ and thus $\frac{\mu(\mathcal{R}(x(m) \cap \alpha))}{\mu(x(m))} = 1$, which is a contradiction with (iii). So $x \notin \mathcal{R}(\alpha)$.
We will define x with induction. Define $x(0)$ to be the code number of the rational segment $(0,1)$. Suppose we have defined $x(0), x(1), \ldots, x(n)$. Split $x(n)$ in two rational segments, $x(n)_0$ and $x(n)_1$, where $x(n)'_0 = x'(n), x(n)''_0 = x(n)'_1 = x'(n) + \frac{1}{2}l(x(n))$ and $x(n)''_1 = x''(n)$. Thus $x(n)_0$ is the first half of $x(n)$ and $x(n)_1$ is the second half of $x(n)$. Find $m \in \mathbb{N}$ such that $\mu(\bar{\alpha}m) \geq (1 - \frac{1}{2^{4n+4}})\mu(\mathcal{R}(\alpha))$. Find $i \in \{0,1\}$ such that $\mu(x(n)_i \cap \bar{\alpha}m) \leq \mu(x(n)_{1-i} \cap \bar{\alpha}m)$. If $i = 0$ then define $x(n+1) = (x(n)'_0 + \frac{1}{2^{2n+2}}l(x(n)), x(n)''_0 + \frac{1}{2^{2n+2}}l(x(n)))$. If $i = 1$ then define $x(n+1) = (x(n)'_1 - \frac{1}{2^{2n+2}}l(x(n)), x(n)''_1 - \frac{1}{2^{2n+2}}l(x(n)))$. We have:

1. $\frac{\mu(x(n) \cap \bar{\alpha}m)}{\mu(x(n))} \leq \frac{\mu(\mathcal{R}(x(n) \cap \alpha))}{\mu(x(n))} \leq 1 - \frac{1}{2^{n+1}}$, and

2. since $\mu(x(n)_i \cap \bar{\alpha}m) \leq \mu(x(n)_{1-i} \cap \bar{\alpha}m)$ we have $\mu(x(n)_i \cap \bar{\alpha}m) \leq \frac{1}{2}\mu(x(n) \cap \bar{\alpha}m)$ and thus $\mu(x(n+1) \cap \bar{\alpha}m) \leq \frac{1}{2}\mu(x(n) \cap \bar{\alpha}m) + \frac{1}{2^{2n+2}}l(x(n))$. Also

3. $\mu(\mathcal{R}(x(n+1) \cap \alpha) \leq \mu(x(n+1) \cap \bar{\alpha}m) + \frac{1}{2^{4n+4}}\mu(\mathcal{R}(\alpha))$, and lastly

4. $\mu(x(n+1)) = \frac{1}{2}\mu(x(n)) = \frac{1}{2}l(x(n)) = \frac{1}{2^{n+1}}$

For the x defined above, properties (i) and (ii) are obvious. The properties $1, 2, 3$ and 4 proof property (iii) as follows:

$$
\begin{aligned}
\frac{\mu(\mathcal{R}(x(n+1)\cap\alpha)}{\mu(x(n+1))} &\leq \frac{\frac{1}{2}\mu(x(n)\cap\bar{\alpha}m)}{\mu(x(n+1))} + \frac{\frac{l(x(n))}{2^{2n+2}}}{\mu(x(n+1))} + \frac{\frac{\mu(\mathcal{R}(\alpha))}{2^{4n+4}}}{\mu(x(n+1))} \text{ (by 3 and 4), so} \\
\frac{\mu(\mathcal{R}(x(n+1)\cap\alpha)}{\mu(x(n+1))} &\leq \frac{\mu(x(n)\cap\bar{\alpha}m)}{\mu(x(n))} + \frac{\frac{l(x(n))}{2^{2n+2}}}{\frac{1}{2^{n+1}}} + \frac{\frac{\mu(\mathcal{R}(\alpha))}{2^{4n+4}}}{\frac{1}{2^{n+1}}} \text{ (by 4), so} \\
\frac{\mu(\mathcal{R}(x(n+1)\cap\alpha)}{\mu(x(n+1))} &\leq 1-\frac{1}{2^{n+1}} + \frac{2^{n+1}l(x(n))}{2^{2n+2}} + \frac{2^{n+1}\mu(\mathcal{R}(\alpha))}{2^{4n+4}} \text{ (by 1), so} \\
\frac{\mu(\mathcal{R}(x(n+1)\cap\alpha)}{\mu(x(n+1))} &\leq 1-\frac{1}{2^{n+1}} + \frac{1}{2^{2n+1}} + \frac{1}{2^{3n+3}} \text{ (since } \mathcal{R}(\alpha)\leq 1\text{), so} \\
\frac{\mu(\mathcal{R}(x(n+1)\cap\alpha)}{\mu(x(n+1))} &\leq 1-\frac{1}{2^{n+2}}
\end{aligned}
$$

□

Corollary 3.2.3. *Suppose X is an almost full subset of $[0,1]$, then there exists an $x \in [0,1]$ such that $x \in X$.*

Proof. Suppose X is an almost full subset of $[0,1]$, then there exists a sequence $X_0, X_1, X_2, \ldots$ of measurable regions which meet the requirements *(i)* and *(ii)* of definition 3.2.1. Pick X_1 from this sequence, then $\mu(X_1) < \frac{1}{2} < 1$. For every $x \in [0,1]$ if $x \notin X_1$ then $x \in X$. By lemma 3.2.2 we can construct an $x \in [0,1]$ such that $x \notin X_1$. □

3.3 Measurable functions and sets

We are now able to define when a function is measurable.

Definition 3.3.1. *A partial bounded function $f : [0,1] \to \mathbb{R}$ is* ***measurable*** *if and only if there exists an infinite sequence $X_0, X_1, X_2, \ldots$ of measurable regions and an infinite sequence $v_0, v_1, v_2, \ldots$ of elementary sets of rectangles such that:*

(i) for every $n \in \mathbb{N}$, $\mu(X_n) < \frac{1}{2^n}$ and $\mathrm{Ar}^*(v_n) < \frac{1}{2^n}$, and

(ii) for every $n \in \mathbb{N}$, every $x \in [0,1]$, whenever $x \notin X_n$ then $x \in \mathrm{dom}(f)$ and v_n captures f.

We define $\int f(x)\,\mathrm{d}x = \lim_{n\to\infty} \underline{I}(v_n) = \lim_{n\to\infty} \overline{I}(v_n)$ to be the integral of f.

Note that the domain of a measurable function is almost full. The following lemma shows that the above definition makes sense.

Lemma 3.3.2. *Suppose $f : [0,1] \to \mathbb{R}$ is a partial bounded function, $X_0, X_1, X_2, \ldots$ is an infinite sequence of measurable regions and $v_0, v_1, v_2, \ldots$ is an infinite sequence of elementary sets of rectangles such that conditions (i) and (ii) of definition 3.3.1 are met. Then $\lim_{n\to\infty} \underline{I}(v_n)$ and $\lim_{n\to\infty} \overline{I}(v_n)$ exist, are equal and do not depend on the choice of the infinite sequence $v_0, v_1, v_2, \ldots$ and the measurable regions $X_0, X_1, X_2, \ldots$*

Proof. We claim the following:

(i) $\lim_{n\to\infty} \underline{I}(v_n)$ and $\lim_{n\to\infty} \overline{I}(v_n)$ exist. That is, for all $k \in \mathbb{N}$ there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|\underline{I}(v_n) - \underline{I}(v_m)| \leq \frac{1}{k}$ and $|\overline{I}(v_n) - \overline{I}(v_m)| \leq \frac{1}{k}$

(ii) $\lim_{n\to\infty} \underline{I}(v_n) = \lim_{n\to\infty} \overline{I}(v_n)$

(iii) For every infinite sequence $Y_0, Y_1, Y_2, \ldots$ of measurable regions and every infinite sequence $u_0, u_1, u_2, \ldots$ of elementary sets of rectangles such that conditions *(i)* and *(ii)* of definition 3.3.1 are met we have: $\lim_{n\to\infty} \overline{I}(v_n) = \lim_{n\to\infty} \overline{I}(w_n)$

This shows that the integral of definition 3.3.1 makes sense. We will now prove (i), (ii) and (iii).

(i) f is a bounded function, thus find $M \in \mathbb{N}$ such that $-M \leq f(x) \leq M$ for all $x \in [0,1]$. Pick $k \in \mathbb{N}$ and find $N \in \mathbb{N}$ such that $\frac{2M+1}{2^{N-1}} \leq \frac{1}{k}$. Pick $n, m \geq N$.
We have the measurable regions X_n and X_m and the elementary sets of rectangles v_n and v_m. For readability we define $v := v_n$, $u := v_m$, $l_n := \text{length}(v_n)$ and $l_m := \text{length}(v_m)$. Also, we define two new measurable regions X and Y. For this, find $p, q \in \mathbb{N}$ such that $\mu(X_n) + \frac{2(l_n+1)}{p} < \frac{1}{2^n}$ and $\mu(X_m) + \frac{2(l_m+1)}{q} < \frac{1}{2^m}$. Now define $X := X_n \cup \{\big((v_i)'_0 - \frac{1}{p}, (v_i)'_0 + \frac{1}{p}\big) \mid 0 \leq i < l_n\} \cup \big((v_{l_n-1})''_0 - \frac{1}{p}, (v_{l_n-1})''_0 + \frac{1}{p}\big)$ and $Y := X_m \cup \{\big((u_i)'_0 - \frac{1}{q}, (u_i)'_0 + \frac{1}{q}\big) \mid 0 \leq i < l_m\} \cup \big((u_{l_m-1})''_0 - \frac{1}{q}, (u_{l_m-1})''_0 + \frac{1}{q}\big)$. Thus, X is the union of X_n and small open intervals around the boundaries of v and Y is the union of X_m and small open intervals around the boundaries of u. Clearly, X and Y are measurable regions. Note: we now have, for all $x \in [0,1]$, if $x \notin X \cup Y$ then there exists $0 \leq i < l_n$ and $0 \leq j < l_m$ such that $x \;\varepsilon_0\; (v_i)_0$ and $x \;\varepsilon_0\; (u_j)_0$.
Next, we will consider the elementary sets of rectangles. The idea is to show that almost every rectangle of v "touches" a rectangle of u. See figure 5. For this, we define $w_{il_m+j} := (v_i)_0 \cap (u_j)_0$ and $W := \{w_{il_m+j} \mid 0 \leq i < l_n, 0 \leq j < l_m \text{ and } w_{il_m+j} \neq \bot\}$, which clearly is a partition of $[0,1]$. We will separate W into two subsets, $W_\top$ and $W_\bot$, where $W_\top$ is the set of rectangles that "touch" and $W_\bot$ is the set of rectangles that do not "touch". So we define $W_\top := \{w_{il_m+j} \mid w_{il_m+j} \in W \mid (v_i)_1 \cap (v_j)_1 \neq \bot\}$ and $W_\bot := \{w_{il_m+j} \mid w_{il_m+j} \in W \mid (v_i)_1 \cap (v_j)_1 = \bot\}$. Now, for all $x \in [0,1]$, whenever $x \notin X \cup Y$ there exist $0 \leq i < l_n$ and $0 \leq j < l_m$ such that $x \;\varepsilon_0\; w_{il_m+j}$ and, by clause *(ii)* of definition 3.3.1, $w_{il_m+j} \in W_\top$. This means, if $w_{il_m+j} \in W_\bot$ then for all $x \;\varepsilon_0 w_{il_m+j}$, $x \in X \cup Y$. Thus $W_\bot \subseteq X \cup Y$. This gives:

$$\mu\Big(\bigcup_{w \in W_\bot} w\Big) \leq \mu(X \cup Y) \leq \mu(X) + \mu(Y) \leq \frac{1}{2^n} + \frac{1}{2^m} \leq \frac{1}{2^{\min(n,m)-1}} \leq \frac{1}{2^{N-1}}$$

Now we can prove the claim:

$$
\begin{aligned}
|\underline{I}(v_n) - \underline{I}(v_m)| &= |\underline{I}(v) - \underline{I}(u)| \\
&= |\sum_{i=0}^{l_n-1} \big((v_i)_0'' - (v_i)_0'\big)(v_i)_1' - \sum_{i=0}^{l_m-1} \big((u_i)_0'' - (u_i)_0'\big)(u_i)_1'| \\
&= |\sum_{w_{il_m+j} \in W} \big(w_{il_m+j}'' - w_{il_m+j}'\big)\big((v_i)_1' - (u_j)_1'\big)| \\
&\leq \sum_{w_{il_m+j} \in W} \big(w_{il_m+j}'' - w_{il_m+j}'\big)|\big((v_i)_1' - (u_j)_1'\big)| \\
&\leq \sum_{w_{il_m+j} \in W_\top} \big(w_{il_m+j}'' - w_{il_m+j}'\big)|\big((v_i)_1' - (u_j)_1'\big)| + \\
&\quad \sum_{w_{il_m+j} \in W_\bot} \big(w_{il_m+j}'' - w_{il_m+j}'\big)|\big((v_i)_1' - (u_j)_1'\big)| \\
&\leq \sum_{w_{il_m+j} \in W_\top} \big(w_{il_m+j}'' - w_{il_m+j}'\big)|\big((v_i)_1'' - (v_i)_1'\big) + \big((u_j)_1'' - (u_j)_1'\big)| + \\
&\quad \sum_{w_{il_m+j} \in W_\bot} \big(w_{il_m+j}'' - w_{il_m+j}'\big)2M \\
&\leq \frac{1}{2^m} + \frac{1}{2^n} + \frac{2M}{2^{N-1}} \leq \frac{2M+1}{2^{N-1}} \leq \frac{1}{k}
\end{aligned}
$$

And similarly we get:

$$
\begin{aligned}
|\overline{I}(v_n) - \overline{I}(v_m)| &= |\overline{I}(v) - \overline{I}(u)| \\
&= |\sum_{i=0}^{l_n-1} \big((v_i)_0'' - (v_i)_0'\big)(v_i)_1'' - \sum_{i=0}^{l_m-1} \big((u_i)_0'' - (u_i)_0'\big)(u_i)_1''| \\
&= |\sum_{w_{il_m+j} \in W} \big(w_{il_m+j}'' - w_{il_m+j}'\big)\big((v_i)_1'' - (u_j)_1''\big)| \\
&\leq \sum_{w_{il_m+j} \in W} \big(w_{il_m+j}'' - w_{il_m+j}'\big)|\big((v_i)_1'' - (u_j)_1''\big)| \\
&\leq \sum_{w_{il_m+j} \in W_\top} \big(w_{il_m+j}'' - w_{il_m+j}'\big)|\big((v_i)_1'' - (u_j)_1''\big)| + \\
&\quad \sum_{w_{il_m+j} \in W_\bot} \big(w_{il_m+j}'' - w_{il_m+j}'\big)|\big((v_i)_1'' - (u_j)_1''\big)| \\
&\leq \sum_{w_{il_m+j} \in W_\top} \big(w_{il_m+j}'' - w_{il_m+j}'\big)|\big((v_i)_1'' - (v_i)_1'\big) + \big((u_j)_1'' - (u_j)_1'\big)| + \\
&\quad \sum_{w_{il_m+j} \in W_\bot} \big(w_{il_m+j}'' - w_{il_m+j}'\big)2M \\
&\leq \frac{1}{2^m} + \frac{1}{2^n} + \frac{2M}{2^{N-1}} \leq \frac{2M+1}{2^{N-1}} \leq \frac{1}{k}
\end{aligned}
$$

(ii) Let, for all $n \in \mathbb{N}$, $l_n = \text{length}(v_n) - 1$.

$$\forall n \in \mathbb{N}\ [\text{Ar}(v_n) \leq \frac{1}{2^n}] \implies$$

$$\forall n \in \mathbb{N}\ [\sum_{i=0}^{l_n} \big(((v_n)_i)''_0 - ((v_n)_i)'_0\big)\big(((v_n)_i)''_1 - ((v_n)_i)'_1\big) \leq \frac{1}{2^n}] \implies$$

$$\forall n \in \mathbb{N}\ [\sum_{i=0}^{l_n} \big(((v_n)_i)''_0 - ((v_n)_i)'_0\big)((v_n)_i)''_1 - \sum_{i=0}^{l_n} \big(((v_n)_i)''_0 - ((v_n)_i)'_0\big)((v_n)_i)'_1 \leq \frac{1}{2^n}] \implies$$

$$\forall n \in \mathbb{N}[\sum_{i=0}^{l_n} \big(((v_n)_i)''_0 - ((v_n)_i)'_0\big)((v_n)_i)''_1 \leq \sum_{i=0}^{l_n} \big(((v_n)_i)''_0 - ((v_n)_i)'_0\big)((v_n)_i)'_1 + \frac{1}{2^n}] \implies$$

$$\lim_{n\to\infty} \Big(\sum_{i=0}^{l_n} \big(((v_n)_i)''_0 - ((v_n)_i)'_0\big)((v_n)_i)''_1\Big) \leq$$

$$\lim_{n\to\infty} \Big(\sum_{i=0}^{l_n} \big(((v_n)_i)''_0 - ((v_n)_i)'_0\big)((v_n)_i)'_1 + \frac{1}{2^n}\Big) =$$

$$\lim_{n\to\infty} \Big(\sum_{i=0}^{l_n} \big(((v_n)_i)''_0 - ((v_n)_i)'_0\big)((v_n)_i)'_1\Big) + \lim_{n\to\infty} \frac{1}{2^n} =$$

$$\lim_{n\to\infty} \Big(\sum_{i=0}^{l_n} \big(((v_n)_i)''_0 - ((v_n)_i)'_0\big)((v_n)_i)'_1\Big)$$

So $\lim_{n\to\infty} \underline{I}(v_n) \geq \lim_{n\to\infty} \overline{I}(v_n)$. Also:

$$\forall n \in \mathbb{N}[(v_n)_i)'_1 \leq (v_n)_i)''_1] \implies$$

$$\forall n \in \mathbb{N}[\sum_{i=0}^{l_n} \big(((v_n)_i)''_0 - ((v_n)_i)'_0\big)((v_n)_i)'_1 \leq \sum_{i=0}^{l_n} \big(((v_n)_i)''_0 - ((v_n)_i)'_0\big)((v_n)_i)''_1] \implies$$

$$\lim_{n\to\infty} \Big(\sum_{i=0}^{l_n} \big(((v_n)_i)''_0 - ((v_n)_i)'_0\big)((v_n)_i)'_1\Big) \leq \lim_{n\to\infty} \Big(\sum_{i=0}^{l_n} \big(((v_n)_i)''_0 - ((v_n)_i)'_0\big)((v_n)_i)''_1\Big)$$

So $\lim_{n\to\infty} \underline{I}(v_n) \leq \lim_{n\to\infty} \overline{I}(v_n)$.

(iii) It is sufficient to prove, for every $k \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|\overline{I}(v_n) - \overline{I}(u_m)| \leq \frac{1}{k}$. Again, f is a bounded function, thus find $M \in \mathbb{N}$ such that $-M \leq f(x) \leq M$ for all $x \in [0, 1]$. Pick $k \in \mathbb{N}$ and find $N \in \mathbb{N}$ such that $\frac{2M+1}{2^{N-1}} \leq \frac{1}{k}$. Pick $n, m \geq N$. This proof is very similar to the proof in (i).
We have the measurable regions X_n and Y_m and the elementary sets of rectangles v_n and u_m. For readability we define $v := v_n$, $u := u_m$, $l_n := \text{length}(v_n)$ and $l_m := \text{length}(u_m)$. Also, we define two new measurable regions X and Y. For this, find $p, q \in \mathbb{N}$ such that $\mu(X_n) + \frac{2(l_n+1)}{p} < \frac{1}{2^n}$ and $\mu(Y_m) + \frac{2(l_m+1)}{q} < \frac{1}{2^m}$. Now define $X := X_n \cup \{\big((v_i)'_0 - \frac{1}{p}, (v_i)'_0 + \frac{1}{p}\big) \mid 0 \leq i < l_n\} \cup \big((v_{l_n-1})''_0 - \frac{1}{p}, (v_{l_n-1})''_0 + \frac{1}{p}\big)$ and $Y := Y_m \cup \{\big((u_i)'_0 - \frac{1}{q}, (u_i)'_0 + \frac{1}{q}\big) \mid 0 \leq i < l_m\} \cup \big((u_{l_m-1})''_0 - \frac{1}{q}, (u_{l_m-1})''_0 + \frac{1}{q}\big)$. Thus, X is the union of X_n and small open intervals around the boundaries of v and Y is the union of Y_m and small open intervals around the boundaries of u. Clearly, X and Y are measurable regions. Note: we now have, for all $x \in [0, 1]$, if $x \notin X \cup Y$ then there exists $0 \leq i < l_n$ and $0 \leq j < l_m$ such that $x\ \varepsilon_0\ (v_i)_0$ and $x\ \varepsilon_0\ (u_j)_0$.

Next, we will consider the elementary sets of rectangles. Again, we will see that almost every rectangle of v "touches" a rectangle of u. For this, we again define $w_{il_m+j} := (v_i)_0 \cap (u_j)_0$ and $W := \{w_{il_m+j} \mid 0 \leq i < l_n, 0 \leq j < l_m \text{ and } w_{il_m+j} \neq \bot\}$, which again is a partition of $[0,1]$. We will again separate W into two subsets, $W_\top := \{w_{il_m+j} \mid w_{il_m+j} \in W \mid (v_i)_1 \cap (v_j)_1 \neq \bot\}$ and $W_\bot := \{w_{il_m+j} \mid w_{il_m+j} \in W \mid (v_i)_1 \cap (v_j)_1 = \bot\}$. Now, for all $x \in [0,1]$, whenever $x \notin X \cup Y$ there exist $0 \leq i < l_n$ and $0 \leq j < l_m$ such that $x \; \varepsilon_0 \; w_{il_m+j}$ and, by clause (ii) of definition 3.3.1, $w_{il_m+j} \in W_\top$. This means, if $w_{il_m+j} \in W_\bot$ then for all $x \; \varepsilon_0 w_{il_m+j}$, $x \in X \cup Y$. Thus $W_\bot \subseteq X \cup Y$. So :

$$\mu(\bigcup_{w \in W_\bot} w) \leq \mu(X \cup Y) \leq \mu(X) + \mu(Y) \leq \frac{1}{2^n} + \frac{1}{2^m} \leq \frac{1}{2^{\min(n,m)-1}} \leq \frac{1}{2^{N-1}}$$

This gives us:

$$\begin{aligned}
|\overline{I}(v_n) - \overline{I}(u_m)| &= |\overline{I}(v) - \overline{I}(u)| \\
&= |\sum_{i=0}^{l_n-1} \big((v_i)''_0 - (v_i)'_0\big)(v_i)''_1 - \sum_{i=0}^{l_m-1} \big((u_i)''_0 - (u_i)'_0\big)(u_i)''_1| \\
&= |\sum_{w_{il_m+j} \in W} \big(w''_{il_m+j} - w'_{il_m+j}\big)\big((v_i)''_1 - (u_j)''_1\big)| \\
&\leq \sum_{w_{il_m+j} \in W} \big(w''_{il_m+j} - w'_{il_m+j}\big)|\big((v_i)''_1 - (u_j)''_1\big)| \\
&\leq \sum_{w_{il_m+j} \in W_\top} \big(w''_{il_m+j} - w'_{il_m+j}\big)|\big((v_i)''_1 - (u_j)''_1\big)| + \\
&\quad \sum_{w_{il_m+j} \in W_\bot} \big(w''_{il_m+j} - w'_{il_m+j}\big)|\big((v_i)''_1 - (u_j)''_1\big)| \\
&\leq \sum_{w_{il_m+j} \in W_\top} \big(w''_{il_m+j} - w'_{il_m+j}\big)|\big((v_i)''_1 - (v_i)'_1\big) + \big((u_j)''_1 - (u_j)'_1\big)| + \\
&\quad \sum_{w_{il_m+j} \in W_\bot} \big(w''_{il_m+j} - w'_{il_m+j}\big)2M \\
&\leq \frac{1}{2^m} + \frac{1}{2^n} + \frac{2M}{2^{N-1}} \leq \frac{2M+1}{2^{N-1}} \leq \frac{1}{k}
\end{aligned}$$

□

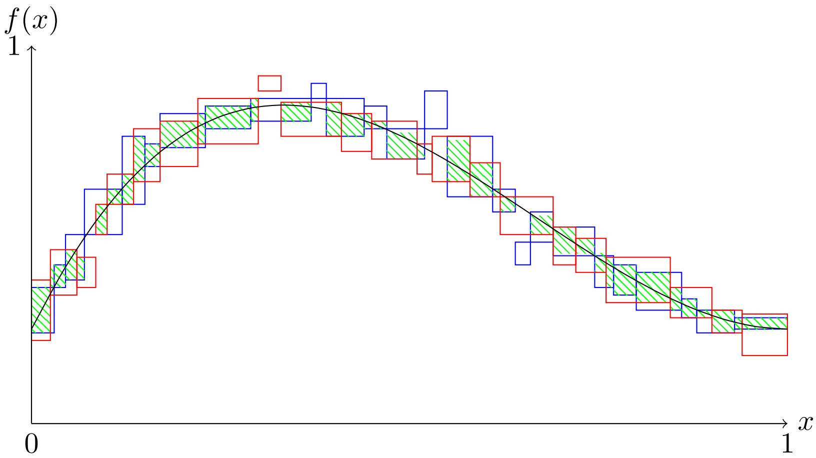

Figure 5: Most elementary rectangles "touch".

We will now prove some important lemmas and theorems about measurable function which will become useful in chapter 5 of this thesis.

Lemma 3.3.3. *Suppose $f : [0,1] \to \mathbb{R}$ is a bounded and measurable function and suppose $g : [0,1] \to \mathbb{R}$ is a bounded function and $f(x) = g(x)$ almost everywhere (that is, there exists an almost full set $Y \subseteq [0,1]$ such that $\forall y \in Y[f(y) = g(y)]$) then g is measurable and $\int f(x)\,\mathrm{d}x = \int g(x)\,\mathrm{d}x$.*

Proof. Since f is a measurable function we know there exists a sequence $X_0, X_1, X_2, \ldots$ of measurable regions and a sequence $v_0, v_1, v_2, \ldots$ of elementary sets of rectangles which meet the requirements *(i)* and *(ii)* of definition 3.3.1.
Also, since there exists an almost full set $Y \subseteq [0,1]$ such that $\forall y \in Y[f(y) = g(y)]$ we know there exists a sequence $Y_0, Y_1, Y_2, \ldots$ of measurable regions which meet the requirements *(i)* and *(ii)* of definition 3.2.1.
To prove that g is measurable we take the sequence $Z_0 = X_0 \cup Y_0, Z_1 = X_1 \cup Y_1, Z_2 = X_2 \cup Y_2, \ldots$ of measurable regions and the sequence $v_0, v_1, v_2, \ldots$ of elementary sets of rectangles. Now we have:

(i) For every $n \in \mathbb{N}$ $[\mathrm{Ar}(v_n) < \frac{1}{2^n}]$ and $\mu(Z_n) \leq \mu(X_n) + \mu(Y_n) < \frac{1}{2^{n-1}}$, and

(ii) For every $n \in \mathbb{N}$ and every $x \in [0,1]$ whenever $x \notin Z_n$) then $x \notin X_n$ and $x \notin Y_n$. This means $x \in Y$ and $x \in \mathrm{dom}(f)$ and so $x \in \mathrm{dom}(g)$. This also means $f(x) = g(x)$ and thus $\forall i < \mathrm{length}(v_n)$ when $x \; \varepsilon_0 \; ((v_n)_i)_0$ then $f(x) \; \varepsilon_0 \; ((v_n)_i)_1$, so $g(x) \; \varepsilon_0 \; ((v_n)_i)_1$.

It immediately follows that $\int f(x)\,\mathrm{d}x = \int g(x)\,\mathrm{d}x$. □

Theorem 3.3.4. *Suppose $f : [0,1] \to \mathbb{R}$ is a partial function such that* $\mathrm{dom}(f)$ *is almost full and such that f is bounded, then f is measurable.*

To prove this theorem we need three lemmas.

Lemma 3.3.5. *Suppose $\alpha = \alpha(0), \alpha(1), \alpha(2), \ldots$ is an infinite sequence of code numbers of rational segments such that the sequence $\mu(\bar{\alpha}1), \mu(\bar{\alpha}2), \ldots$ converges. Suppose $a \in S$. Then there exists $n \in \mathbb{N}$ such that $\frac{\mu(\bar{\alpha}n \cap a)}{\mu(a)} \geq \frac{1}{2}$ or there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $\frac{\mu(\bar{\alpha}m \cap a)}{\mu(a)} < 1 - \frac{1}{2^n}$.*

Proof. Suppose $\alpha = \alpha(0), \alpha(1), \alpha(2), \ldots$ is an infinite sequence of code numbers of rational segments such that the sequence $\mu(\bar{\alpha}1), \mu(\bar{\alpha}2), \ldots$ converges. Suppose $a \in S$. Find $k \in \mathbb{N}$ such that $\mu(\alpha) - \mu(\bar{\alpha}k) < \frac{1}{2}\mu(a)$. Suppose $\frac{\mu(\bar{\alpha}k \cap a)}{\mu(a)} \geq \frac{1}{2}$, then we are done. So suppose not $\frac{\mu(\bar{\alpha}k \cap a)}{\mu(a)} \geq \frac{1}{2}$ then, since these are rational numbers, $\mu(\bar{\alpha}k \cap a) \leq \frac{1}{2}\mu(a)$. Pick $m \in \mathbb{N}$ and suppose $\mu(\bar{\alpha}m \cap a) = \mu(a)$. This means $m > k$ and more then half of a gets covered by $\big(\alpha(m-1) \cup \cdots \cup \alpha(k)\big) \setminus \big(\alpha(k-1) \cup \cdots \cup \alpha(0)\big)$. But $\mu\Big(\big(\alpha(m-1) \cup \cdots \cup \alpha(k)\big) \setminus \big(\alpha(k-1) \cup \cdots \cup \alpha(0)\big)\Big) = \sum_{n=k}^{m-1} \mu\big(\alpha(n) \setminus \bigcup_{i=0}^{n-1} \alpha(i)\big) = \mu(\bar{\alpha}m) - \mu(\bar{\alpha}k) < \mu(\alpha) - \mu(\bar{\alpha}k) < \dfrac{1}{2}\mu(a)$, which is a contradiction. So, for every $m \in \mathbb{N}$, $\mu(\bar{\alpha}m \cap a) < \mu(a)$. □

Lemma 3.3.6. *Suppose $\alpha = \alpha(0), \alpha(1), \alpha(2), \ldots$ is an infinite sequence of code numbers of rational segments such that the sequence $\mu(\bar{\alpha}1), \mu(\bar{\alpha}2), \ldots$ converges. Suppose $a \in S$ and there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $\frac{\mu(\bar{\alpha}m \cap a)}{\mu(a)} < 1 - \frac{1}{2^n}$. Now define $a_0 = (a', \frac{a'+a''}{2})$ and*

$a_1 = (\frac{a'+a''}{2}, a'')$. *Then there exists* $n \in \mathbb{N}$ *such that for all* $m \in \mathbb{N}$, $\frac{\mu(\bar{\alpha}m \cap a_0)}{\mu(a_0)} < 1 - \frac{1}{2^n}$ *or there exists* $n \in \mathbb{N}$ *such that for all* $m \in \mathbb{N}$, $\frac{\mu(\bar{\alpha}m \cap a_1)}{\mu(a_1)} < 1 - \frac{1}{2^n}$.

Proof. Suppose $\alpha = \alpha(0), \alpha(1), \alpha(2), \dots$ is an infinite sequence of code numbers of rational segments such that the sequence $\mu(\bar{\alpha}1), \mu(\bar{\alpha}2), \dots$ converges. Suppose $a \in S$ and there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $\frac{\mu(\bar{\alpha}m \cap a)}{\mu(a)} < 1 - \frac{1}{2^n}$. Find this $n \in \mathbb{N}$ and find $k \in \mathbb{N}$ such that $\mu(\alpha) - \mu(\bar{\alpha}k) < \frac{1}{2^n}\mu(a)$. Find $i \in \{0, 1\}$ such that $\mu(a_i \cap \bar{\alpha}k) \leq \mu(a_{1-i} \cap \bar{\alpha}k)$. Then $\mu(a_i \cap \bar{\alpha}k) \leq \frac{1}{2}\mu(a \cap \bar{\alpha}k)$. Then $\frac{\mu(a_i \cap \bar{\alpha}k)}{\mu(a_i)} \leq \frac{\frac{1}{2}\mu(a \cap \bar{\alpha}k)}{\frac{1}{2}\mu(a)} < 1 - \frac{1}{2^n}$. Also, pick $m \in \mathbb{N}$ and suppose $\frac{\mu(a_i \cap \bar{\alpha}m)}{\mu(a_i)} \geq 1 - \frac{1}{2^{n+1}}$. This means $m > k$ and more then $\frac{1}{2^{n+1}}\mu(a_i)$ of a_i gets covered by $\big(\alpha(m-1) \cup \dots \cup \alpha(k)\big) \setminus \big(\alpha(k-1) \cup \dots \cup \alpha(0)\big)$. But $\mu\Big(\big(\alpha(m-1) \cup \dots \cup \alpha(k)\big) \setminus \big(\alpha(k-1) \cup \dots \cup \alpha(0)\big)\Big) = \sum_{n=k}^{m-1} \mu\big(\alpha(n) \setminus \bigcup_{i=0}^{n-1} \alpha(i)\big) = \mu(\bar{\alpha}m) - \mu(\bar{\alpha}k) < \mu(\alpha) - \mu(\bar{\alpha}k) < \frac{1}{2^{n+1}}\mu(a) = \frac{1}{2^{n+1}}\mu(a_i)$, which is a contradiction. So, for every $m \in \mathbb{N}$, $\frac{\mu(a_i \cap \bar{\alpha}m)}{\mu(a_i)} < 1 - \frac{1}{2^{n+1}}$. □

Lemma 3.3.7. *Suppose* $X \subseteq [0, 1]$ *and* X *is almost full. Then for every* $n \in \mathbb{N}$ *we can find a measurable region* Y *and a fan* τ *such that* $\mu(Y) < \frac{1}{2^n}$, $\tau \subseteq X$ *and for every* $x \in [0, 1]$ *if* $\neg(x \in Y)$ *then* $x \in \tau$.

Proof. X is almost full, so find a measurable region X_n such that $\mu(X_n) < \frac{1}{2^{n+2}}$ and for every $x \in [0, 1]$ if $\neg(x \in X_n)$ then $x \in X$. Since X_n is a measurable region we have an infinite sequence $\alpha(0), \alpha(1), \alpha(2), \dots$ of code numbers of rational segments such that $\mu(\bar{\alpha}1), \mu(\bar{\alpha}2)$ converges and $\mathcal{R}(\alpha) = X_n$.
We define, for every $n \in \mathbb{N}$ a set of rational segments $S_n := \{(0, \frac{1}{2^n}), \dots, (\frac{2^n-1}{2^n}, 1)\}$. Also, for every $n \in \mathbb{N}$ we define a $R_n \subseteq S_n$ such that:

(i) $R_0 = \{(0, 1)\}$

(ii) For every $n \in \mathbb{N}$ and $a \in R_n$ we can find a $b \in R_{n+1}$ such that $b \sqsubset a$

(iii) For every $n \in \mathbb{N}$ and $a \in S_n$, if $a \in R_n$ then there exists $k \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $\frac{\mu(\bar{\alpha}k \cap a)}{\mu(a)} < 1 - \frac{1}{2^k}$ and if $a \notin R_n$ then we can find $b \in S$ such that $a \sqsubset b$ and $\frac{\mu(b \cap \alpha)}{\mu(b)} \geq \frac{1}{2}$.

We can always make R_n by lemma 3.3.6 and 3.3.5. We now define a sequence $\beta = \beta(0), \beta(1), \dots$ of rational intervals. This sequence is created by first numbering the rational intervals of $S_1 \setminus R_1$, then of $\{a \in S_2 | \exists b \in R_1[a \sqsubset b]\} \setminus R_2$, then of $\{a \in S_3 | \exists b \in R_2[a \sqsubset b]\} \setminus R_3$, etc. We claim that the set $Y' = \mathcal{R}(\beta)$ is a measurable region. Thus we have to show that the sequence $\mu(\bar{\beta}1), \mu(\bar{\beta}2), \dots$ converges. Suppose $n \in mathbbN$. Find $N \in \mathbb{N}$ such that $\mu(\alpha) - \mu(\bar{\alpha}N) < \frac{1}{2^n 4}$. Find $M \in \mathbb{N}$ such that $\mu(\{a \in S_M | a \text{ is covered by } \bar{\alpha}N\} \geq \mu(\bar{\alpha}N) - \frac{1}{2^n 4}$. Now find $L \in \mathbb{N}$ such that $\{a \in S_M | a \text{ is covered by } \bar{\alpha}N\} \subseteq \{\beta(0), \dots, \beta(L-1)\}$. Then we have, for every $k \geq L$, $\mu(\bar{\beta}L) \leq \mu(\bar{\beta}k) \leq \mu(\bar{\beta}L) + \frac{2}{2^n 4} + \frac{2}{2^n 4} = \mu(\bar{\beta}L) + \frac{1}{2^n}$. Where the last inequality holds since $\mu(\bar{\beta}L) \geq \mu(\{a \in S_M | a \text{ is covered by } \bar{\alpha}N\}) \geq \mu(\bar{\alpha}N) - \frac{1}{2^n 4} \geq \mu(\alpha) - \frac{1}{2^n 4} - \frac{1}{2^n 4}$. This means $\beta(L), \beta(L+1), \dots$ can cover only $\frac{2}{2^n 4}$ of α extra. Since β consists of intervals which are not in R_n this means these segments are covered for more then half by α. Thus the measure of Y' can only grow $2\frac{2}{2^n 4}$. For the same reasoning, $\mu(Y') \leq 2\mu(\alpha)$.
Now define τ as follows:

$$\alpha \in \tau \iff \forall n \in \mathbb{N}[\alpha(n) \in \lambda_{n+1} \wedge \alpha(n+1) \sqsubseteq \alpha(n) \wedge \alpha(n) \in R_n]$$

Obviously, τ is a spread.
Furthermore we can show for every $x \in [0, 1]$ if $\neg(x \in Y')$ and for every $q \in \mathbb{Q}$, $q\#x$ then there

exists $\gamma \in \tau$ such that $\gamma \equiv x$. Pick $x \in [0,1]$ and suppose $\neg(x \in Y')$ and for every $q \in \mathbb{Q}$, $q \# x$. We define $\gamma = \gamma(0), \gamma(1), \gamma(2), \ldots$ with induction. Define $\gamma(0) = [0,1]$. Now suppose $\gamma(0), \ldots, \gamma(n)$ are defined in such a way that for all $k \leq n[\gamma(k) \in \lambda_{k+1} \wedge \gamma(k) \sqsubseteq \gamma(k-1) \wedge \gamma(k) \in R_k \wedge \exists m \in \mathbb{N}[x(m) \sqsubseteq \gamma(k)]$ and suppose $\gamma(n) = [\frac{a}{2^n}, \frac{a+1}{2^n}]$ with $0 \leq a < 2^n$. Since $\frac{2a+1}{2^{n+1}} \in \mathbb{Q}$ we can find $k \in \mathbb{N}$ such that $\frac{2a+1}{2^{n+1}} > x''(k)$ or $\frac{2a+1}{2^{n+1}} < x'(k)$. If $\frac{2a+1}{2^{n+1}} > x''(k)$ define $\gamma(n+1) = [\frac{2a}{2^{n+1}}, \frac{2a+1}{2^{n+1}}]$ else define $\gamma(n+1) = [\frac{2a+1}{2^{n+1}}, \frac{2a+2}{2^{n+1}}]$. Now obviously, $\gamma(n+1) \in \lambda_{n+2}$, $\gamma(n+1) \sqsubseteq \gamma(n)$. Also, find $m \in \mathbb{N}$ such that $x(m) \sqsubseteq \gamma(n)$. This means $\gamma(n)' \leq x'(m)$ and $x''(m) \leq \gamma(n)''$. Define $l = \max(k, m)$. Suppose $\gamma(n+1) = [\frac{2a}{2^{n+1}}, \frac{2a+1}{2^{n+1}}]$ then $\gamma(n+1)' = \gamma(n)' \leq x'(m) < x'(l)$ and $\gamma(n+1)'' > x''(m) > x''(l)$ so $x(l) \sqsubseteq \gamma(n+1)$. Suppose $\gamma(n+1) = [\frac{2a+1}{2^{n+1}}, \frac{2a+2}{2^{n+1}}]$ then $\gamma(n+1)' < x'(k) < x'(l)$ and $\gamma(n+1)'' = \gamma(n)'' \geq x''(k) > x''(l)$ so $x(l) \sqsubseteq \gamma(n+1)$. Lastly, suppose $\gamma(n+1) \notin R_{n+1}$ then there exists $m \in \mathbb{N}$ such that $\gamma(n+1) = \beta(m)$. But, since $x \notin Y'$, $\neq (\beta(m)' < x'(l) < x''(l) < \beta(m)'')$ so $\neq (\gamma(n+1)' < x'(l) < x''(l) < \gamma(n+1)'')$ which is a contradiction. Thus $\gamma(n+1) \in R_{n+1}$.
Furthermore, the set $B = \{x \in [0,1] | \forall q \in \mathbb{Q}[q \# x]\}$ is almost full, so we can find a measurable region Y such that $\mu(Y) \leq 2\mu(Y')$ and such that for all $x \in [0,1]$ if $x \in Y'$ then $x \in Y$ and if $x \notin Y$ then $x \in B$. So, for all $x \in [0,1]$ if $x \notin Y$ then there exists $\gamma \in \tau$ such that $\gamma \equiv x$. □

We can now prove theorem 3.3.4.

Proof of Theorem 3.3.4. Suppose $f : [0,1] \to \mathbb{R}$ is a partial function such that $\text{dom}(f)$ is almost full and such that f is bounded. Find $M \in \mathbb{N}$ such that for all $x \in [0,1]$, $-M \leq f(x) \leq M$. Pick $n \in \mathbb{N}$. We want to find a measurable region X_n and an elementary set of rectangles v_n such that:

(i) $\mu(X_n) \leq \frac{1}{2^n}$ and $\text{Ar}(v_n) \leq \frac{1}{2^n}$

(ii) for every $x \in [0,1]$ if $x \notin X_n$ then $x \in \text{dom}(f)$ and v_n captures f.

By lemma 3.3.7, we can find a measurable region X_n and a fan $\tau \subseteq \text{dom}(f)$ such that $\mu(X_n) \leq \frac{1}{2^n}$ and for all $x \in [0,1]$ if $x \notin Y$ then $x \in \tau$. Find $k \in \mathbb{N}$ such that $\frac{1}{2^k} + \frac{M}{2^k} \leq \frac{1}{2^n}$. By the continuity theorem f is continuous. Also $\tau \subseteq \text{dom}(f)$, so for every $\gamma \in \tau$ there exists $y \in [0,1]$ such that $f(\gamma) = y$. This means, for every $\gamma \in \tau$ there exists $m \in \mathbb{N}$ such that for all $x \in [0,1]$ if $|x - \gamma| \leq \frac{1}{2^m}$ then $|f(x) - f(\gamma)| \leq \frac{1}{2^k}$. This means, for every $\gamma \in \tau$ there exists $v \in R$ such that $\text{length}(v_0) \leq \frac{1}{2^m}$, $\text{length}(v_1) \leq \frac{1}{2^k}$, $\gamma \;\varepsilon_0\; v_0$ and for all $x \in [0,1]$ if $x \;\varepsilon_0\; v_0$ then $f(x) \;\varepsilon_0\; v_1$. Since τ is a fan we can use the fan theorem and find $N \in \mathbb{N}$ such that for every $\gamma \in \tau$ there exists $v \in R$ with $v \leq N$ such that $\text{length}(v_0) \leq \frac{1}{2^m}$, $\text{length}(v_1) \leq \frac{1}{2^k}$, $\gamma \;\varepsilon_0\; v_0$ and for all $x \in [0,1]$ if $x \;\varepsilon_0\; v_0$ then $f(x) \;\varepsilon_0\; v_1$. This means we can construct a set of rectangles v with $\text{Ar}(v) \leq \frac{1}{2^k}$ such that v captures $f \upharpoonright \tau$. We define $v_n = v \cup \{a \in R | a_0 \text{ is not covered by } v \text{ and } a_1 = [-M, M]\}$. Then $\text{Ar}(v_n) \leq \frac{1}{2^k} + \frac{M}{2^k} \leq \frac{1}{2^n}$ and for every $x \in [0,1]$ if $x \notin Y$ then $x \in \tau$ so $x \in \text{dom}(f)$ and v_n captures f. □

We will now define when a set X is measurable. For this we need the definition of a characteristic function.

Definition 3.3.8. *The characteristic function of a set X is defined as follows:*

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } \neg(x \in X) \end{cases}$$

Definition 3.3.9. *Suppose $X \subseteq [0,1]$. X is measurable if and only if its characteristic function χ_X is. Assume $X \subseteq [0,1]$ is measurable, then the measure of X is $\mu(X) := \int \chi_X(x)\, \mathrm{d}x$.*

Lemma 3.3.10. *Suppose X is a measurable set, then $X \cup ([0,1] \setminus X)$ is almost full.*

Proof. Since X is a measurable set, its characteristic function χ_X is measurable. As already noted after 3.3.1, the domain of a measurable function is almost full. We have $\mathrm{dom}(\chi_X) = X \cup ([0,1] \setminus X)$. Since χ_X is measurable, there exists a sequence $X_0, X_1, X_2, \dots$ of measurable regions and a sequence $v_0, v_1, v_2, \dots$ of elementary sets of rectangles such that they meet the requirements (i) and (ii) of definition 3.3.1. This means for every $n \in \mathbb{N}$ and every $x \in [0,1]$ if $x \notin X_n$ then $x \in \mathrm{dom}(\chi_X) = X \cup ([0,1] \setminus X)$. □

We can prove an even stronger lemma then corollary 3.2.3. We will not prove it here but refer to [6] for a proof.

Lemma 3.3.11. *Suppose X is a measurable set and $\mu(X) > 0$ then there exists $x \in X$ such that $x \# ([0,1] \setminus X)$.*

We will now prove a number of lemmas about measurable sets.

Lemma 3.3.12. *Suppose X is a measurable set and $X = X' \cup X''$ such that $X' \cap X'' = \emptyset$. Then X' and X'' are measurable.*

Proof. Suppose X is a measurable set and $X = X' \cup X''$ such that $X' \cap X'' = \emptyset$. Since X is measurable we know, by lemma 3.3.10, $X \cup ([0,1] \setminus X)$ is almost full. This means $X' \cup X'' \cup ([0,1] \setminus (X' \cup X''))$ is almost full. But, since $X \cap X'' = \emptyset$ we know $X'' \subseteq ([0,1] \setminus X')$ so $X' \cup X'' \cup ([0,1] \setminus (X' \cup X'')) \subseteq X' \cup ([0,1] \setminus X')$ which means $X' \cup ([0,1] \setminus X')$ is almost full. By theorem 3.3.4 this means $\chi_{X'}$ is measurable thus X' is measurable. In a similar way we prove X'' is measurable. □

Lemma 3.3.13. *Suppose X is a measurable set. Suppose $a \in [0,1]$ and $a \equiv 0$ or $a \# 0$. Then $Xa := \{\ x \in [0,1] \mid x = ya \mid y \in X\}$ is measurable and $\mu(Xa) = a\mu(X)$. Also if there exists a region $v = (v', v'')$ and a region $w = (w', w'')$ such that $v' < 0 < v''$ and $w' < 1 < w''$ such that for each $x \in v$ we know $x \notin X$ and for each $x \in W$ we know $x \notin X$ then $X + a := \{\ x \mid x = y + a \mid y \in X\}$ is measurable for all $a \in \mathbb{R}$ if and only if $X + a \subseteq [0,1]$. We then have $\mu(X + a) = \mu(X)$.*

Proof. Suppose X is measurable, then there exists an infinite sequence of measurable regions $X_0, X_1, X_2, \dots$ and in infinite sequence of elementary sets of rectangles $v_0, v_1, v_2, \dots$ such that the requirements (i) and (ii) of definition 3.3.1 are met. Pick any $a \in [0,1]$, then we can prove Xa is measurable with $aX_0, aX_1, aX_2, \dots$ and $av_0, av_1, av_2, \dots$. Here for every $n \in \mathbb{N}$, $aX_n = \mathcal{R}(\beta(0), \beta(1), \beta(2), \dots$ with $\beta(n) = (a\alpha(n)', a\alpha(n)'')$ when $X_n = \mathcal{R}(\alpha(0), \alpha(1), \alpha(2), \dots)$. Also if $a \equiv 0$ then for every $n \in \mathbb{N}$ and every $i < \mathrm{length}(v_n)$ if $i < \mathrm{length}(v_n) - 1$ then $((av_n)_i)_0 = (0,0)$, if $i = \mathrm{length}(v_n) - 1$ then $((av_n)_i)_0 = (0,1)$ and $((av_n)_i)_1 = ((v_n)_i)_1$. If $a \# 0$ then for every $i < \mathrm{length}(v_n)$, $((av_n)_i)_0 = (a((v_n)_i)_0', a((v_n)_i)_0'')$ and $((av_n)_i)_1 = ((v_n)_i)_1$. Now it is easy to show $\mu(aX) = a\mu(X)$.
Suppose X is such that there exists a region $v = (v', v'')$ and a region $w = (w', w'')$ such that $v' < 0 < v''$ and $w' < 1 < w''$ such that for each $x \in v$ we know $x \notin X$ and for each $x \in W$ we know $x \notin X$. Pick any $a \in [0,1]$ such that $X + a \subseteq [0,1]$ then we can prove $X + a$ is measurable with $a + X_0, a + X_1, a + X_2, \dots$ and $a + v_0, a + v_1, a + v_2, \dots$. Here $a + X_n = \mathcal{R}(\beta(0), \beta(1), \beta(2), \dots$ with $\beta(n) = (a + \alpha(n)', a + \alpha(n)'')$ when $X_n = \mathcal{R}(\alpha(0), \alpha(1), \alpha(2), \dots)$ and:

(i) $((a + v_n)_0)_0 = (0, a + ((v_n)_0)_0')$ and $((a + v_n)_0)_1 = (0,0)$

(ii) for all $0 < i < \mathrm{length}(v_n)$, $((a + v_n)_i)_0 = (a + ((v_n)_{i-1})_0', a + ((v_n)_{i-1})_0'')$ if and only if $a + ((v_n)_{i-1})_0', a + ((v_n)_{i-1})_0'' \leq 1$. If $a + ((v_n)_{i-1})_0'' > 1$ then $((a + v_n)_i)_0 = (a + ((v_n)_{i-1})_0', 1)$ and if $a + ((v_n)_{i-1})_0', a + ((v_n)_{i-1})_0'' > 1$ then $((a + v_n)_i)_0 = (a + ((v_n)_{i-1})_0', a + ((v_n)_{i-1})_0'') = (1,1)$.

(iii) for all $0 < i < \text{length}(v_n)$, $((a+v_n)_i)_1 = ((v_n)_i)_1$

Now it is easy to show $\mu(a+X) = \mu(X)$. $\square$

Lemma 3.3.14. *Suppose X and Y are measurable sets, then $\mu(X \setminus Y) \geq \mu(X) - \mu(Y)$.*

Proof. Suppose X and Y are measurable sets. Then χ_X is measurable so we have an infinite sequence $X_1, X_2, X_3, \ldots$ of measurable regions and an infinite sequence $v_1, v_2, v_3, \ldots$ of elementary sets of rectangles such that the condition of definition 3.3.1 hold. Also, χ_Y is measurable so we have an infinite sequence $Y_1, Y_2, Y_3, \ldots$ of measurable regions and an infinite sequence $u_1, u_2, u_3, \ldots$ of elementary sets of rectangles such that the condition of definition 3.3.1 hold. Now consider $Z_1 = X_1 \cup Y_1, X_2 \cup Y_2, X_3 \cup Y_3, \ldots$. Pick $n \in \mathbb{N}$ and suppose $x \notin Z_n$ then $x \notin X_n$ and $x \notin Y_n$ so $x \in X \cup ([0,1] \setminus X))$ and $x \in Y \cup ([0,1] \setminus Y)$. Suppose $x \in X$ and $x \in Y$ then $x \in ([0,1] \setminus (X \setminus Y))$. Suppose $x \in X$ and $x \in [0,1] \setminus Y$ then $x \in (X \setminus Y)$. Suppose $x \in ([0,1] \setminus X)$ then $x \in ([0,1] \setminus (X \setminus Y))$. So if $x \notin Z_n$ then $x \in (X \setminus Y) \cup ([0,1] \setminus (X \setminus Y)) = \text{dom}(\chi_{X \setminus Y})$. Also $\mu(Z_n) \leq \frac{2}{2^n}$.

We define an elementary set of rectangles w_n for every $n \in \mathbb{N}$. Pick $n \in \mathbb{N}$ and define $l_n = \text{length}(v_n)$ and $k_n = \text{length}(u_n)$. Define, for every $i < l_n$ and $j < k_n$, $((w_n)_{ik_n+j})_0 = (w_{i,j})_0 = ((v_n)_i)_0 \cap ((u_n)_j)_0$ and consider $W = \{(w_{i,j})_0 | 0 \leq i < l_n, 0 \leq j \leq k_n \text{ and } (w_{i,j})_0 \neq \bot\}$ Define $(w_{i,j})_1 = \big(((v_n)_i)_1'\big(1 - ((u_n)_j)_1''\big), ((v_n)_i)_1''\big(1 - ((u_n)_j)_1'\big)\big)$ for every $(w_{i,j})_0 \in W$. Also:

$$
\begin{aligned}
\text{Ar}^*(w_n) &= \sum_{(w_{i,j})_0 \in W} \big((w_{i,j})_1'' - (w_{i,j})_1'\big)\big((w_{i,j})_0'' - (w_{i,j})_0'\big) \\
&= \sum_{(w_{i,j})_0 \in W} \big(((v_n)_i)_1'\big(1 - ((u_n)_j)_1''\big) - ((v_n)_i)_1''\big(1 - ((u_n)_j)_1'\big)\big)\big((w_{i,j})_0'' - (w_{i,j})_0'\big) \\
&= \sum_{(w_{i,j})_0 \in W} \big(((v_n)_i)_1'' - ((v_n)_i)_1'\big)\big((w_{i,j})_0'' - (w_{i,j})_0'\big) \\
&+ \sum_{(w_{i,j})_0 \in W} \big(((v_n)_i)_1'((u_n)_j)_1'' - ((v_n)_i)_1''(u_n)_j)_1'\big((w_{i,j})_0'' - (w_{i,j})_0'\big) \\
&= \sum_{0 \leq i < l_n} \big(((v_n)_i)_1'' - ((v_n)_i)_1'\big)\big((v_n)_i)_0'' - ((v_n)_i)_0'\big) \\
&+ \sum_{0 \leq j < k_n} \big(((v_n)_i)_1'((u_n)_j)_1'' - ((v_n)_i)_1''(u_n)_j)_1'\big(((u_n)_j)_0'' - ((u_n)_j)_0'\big) \\
&\leq \frac{1}{n} + \frac{1}{n}
\end{aligned}
$$

Here the last inequality holds since $((v_n)_i)_1' \leq ((v_n)_i)_1''$ and $0 \leq ((u_n)_j)_1' \leq ((u_n)_j)_1'' \leq 1$.

Now pick $x \in [0,1]$ and $n \in \mathbb{N}$ and suppose $x \notin Z_n$, then $x \in (X \setminus Y)$ or $x \in ([0,1] \setminus (X \setminus Y))$. First suppose $x \in (X \setminus Y)$ then if $x \; \varepsilon_0 \; (w_{i,j})_0$ then $x \; \varepsilon_0 \; ((v_n)_i)_0$ and $x \; \varepsilon_0 \; ((u_n)_j)_0$. So $\chi_X(x) = 1 \; \varepsilon_0 \; ((v_n)_i)_0$ and $\chi_Y(x) = 0 \; \varepsilon_0 \; ((u_n)_j)_0$. This means $((v_n)_i)_1' \leq 1 = ((v_n)_i)_1''$ and $((u_n)_j)_1' = 0 \leq ((u_n)_j)_1'' \leq 1$ so $((w_{i,j})_1' \leq ((v_n)_i)_1' \leq 1$ and $((w_{i,j})_1'' = ((v_n)_i)_1'' = 1$. So $(w_{i,j})_1' \leq \chi_{X \cup Y} = 1 \leq ((w_{i,j})_1''$. Now suppose $x \in ([0,1] \setminus (X \setminus Y))$ then if $x \; \varepsilon_0 \; (w_{i,j})_0$ then $x \; \varepsilon_0 \; ((v_n)_i)_0$ and $x \; \varepsilon_0 \; ((u_n)_j)_0$. So $\chi_X(x) = 1 \; \varepsilon_0 \; ((v_n)_i)_0$ and $\chi_Y(x) = 1 \; \varepsilon_0 \; ((u_n)_j)_0$ or $\chi_X(x) = 0 \; \varepsilon_0 \; ((v_n)_i)_0$. This means $((v_n)_i)_1' \leq 1 = ((v_n)_i)_1''$ and $((u_n)_j)_1' \leq 1 = ((u_n)_j)_1''$ so $((w_{i,j})_1' = 0$ and $((w_{i,j})_1'' \leq 1$. So $((w_{i,j})_1' \leq \chi_{X \cup Y} = 0 \leq ((w_{i,j})_1''$. Or $((v_n)_i)_1' = 0 \leq ((v_n)_i)_1''$ so $((w_{i,j})_1' = 0$ and $((w_{i,j})_1'' \leq 1$. So $((w_{i,j})_1' \leq \chi_{X \cup Y} = 0 \leq ((w_{i,j})_1''$. This means $X \setminus Y$ is measurable. Also:

$$\begin{aligned}
\mu(X \setminus Y) &= \lim_{n\to\infty} \Big(\sum_{\substack{0\le i<l_n \\ 0\le j<k_n}} \big((w_{i,j})''_0 - (w_{i,j})''_0\big)\big((v_i)''_1(1-(u_j)'_1)\big) \\
&= \lim_{n\to\infty} \Big(\sum_{\substack{0\le i<l_n \\ 0\le j<k_n}} \big((w_{i,j})''_0 - (w_{i,j})''_0\big)(v_i)''_1 - \sum_{\substack{0\le i<l_n \\ 0\le j<k_n}} \big((w_{i,j})''_0 - (w_{i,j})''_0\big)(v_i)''_1(u_j)'_1\Big) \\
&= \lim_{n\to\infty} \Big(\sum_{\substack{0\le i<l_n \\ 0\le j<k_n}} \big((w_{i,j})''_0 - (w_{i,j})''_0\big)(v_i)''_1\Big) - \lim_{n\to\infty} \Big(\sum_{\substack{0\le i<l_n \\ 0\le j<k_n}} \big((w_{i,j})''_0 - (w_{i,j})''_0\big)(v_i)''_1(u_j)'_1\Big) \\
&\geq \mu(X) - \mu(Y)
\end{aligned}$$

□

Theorem 3.3.15. *If a set X is measurable and $\mu(X) = k$, then the complement of X, $[0,1] \setminus X$ is measurable and $\mu([0,1] \setminus X) = 1 - k$.*

Proof. The proof will consist of two parts. In part 1 we will prove that $[0,1] \setminus X$ is measurable and in part 2 we will prove $\mu([0,1] \setminus X) = 1 - k$.

1. We know X is measurable, so the characteristic function χ_X is. This means there exists a sequence $X_0, X_1, X_2, \ldots$ of measurable regions and a sequence $v_0, v_1, v_2, \ldots$ of elementary sets of rectangles such that the requirements (i) and (ii) of definition 3.3.1 are met.
   To prove that $\chi_{[0,1]\setminus X}$ is measurable we take the same sequence $X_0, X_1, X_2, \ldots$ of measurable regions. Furthermore we define a sequence $w_0, w_1, w_2, \ldots$ of elementary sets of rectangles as follows. For all $n \in \mathbb{N}$ and for all $i <$ length(v_n) define $((w_n)_i)_0 = ((v_n)_i)_0$, $((w_n)_i)'_1 = 1 - ((v_n)_i)''_1$ and $((w_n)_i)''_1 = 1 - ((v_n)_i)'_1$. We have $\mathrm{Ar}(w_n) = \mathrm{Ar}(v_n) < \frac{1}{2^n}$.
   Now suppose $x \notin X_n$ then $x \in \mathrm{dom}(\chi_X) = \mathrm{dom}(\chi_{[0,1]\setminus X})$. This means either $x \in X$ or $x \in [0,1] \setminus X$.
   First suppose $x \in X$, then $\chi_X(x) = 1$ and $\chi_{[0,1]\setminus X}(x) = 0$. Suppose $x \ \varepsilon_0 \ ((w_n)_i)_0$, then $x \ \varepsilon_0 \ ((v_n)_i)_0$ and thus $1 \ \varepsilon_0 \ ((v_n)_i)_1$. This gives us $((v_n)_i)'_1 \leq 1 \leq ((v_n)_i)''_1$, so $-((v_n)_i)''_1 \leq -1 \leq -((v_n)_i)'_1$ which means $((w_n)_i)'_1 \leq 0 \leq ((w_n)_i)''_1$ and thus $0 \ \varepsilon_0 \ ((w_n)_i)_1$.
   Now suppose $x \in [0,1] \setminus X$, then $\chi_X(x) = 0$ and $\chi_{[0,1]\setminus X}(x) = 1$. Suppose $x \ \varepsilon_0 \ ((w_n)_i)_0$, then $x \ \varepsilon_0 \ ((v_n)_i)_0$ and thus $0 \ \varepsilon_0 \ ((v_n)_i)_1$. This gives us $((v_n)_i)'_1 \leq 0 \leq ((v_n)_i)''_1$, so $-((v_n)_i)''_1 \leq 0 \leq -((v_n)_i)'_1$ which means $((w_n)_i)'_1 \leq 1 \leq ((w_n)_i)''_1$ and thus $1 \ \varepsilon_0 \ ((w_n)_i)_1$.
   So, $\chi_{[0,1]\setminus X}$ is a measurable function.

2. Consider the function $g : [0,1] \to \mathbb{R}$ with $g(x) = \chi_X(x) + \chi_{[0,1]\setminus X}(x)$. For every $x \in \mathrm{dom}(g)$ we have $g(x) = 1$. We will prove that this function is measurable and that $\int \chi_X(x)\,\mathrm{d}x + \int \chi_{[0,1]\setminus X}(x)\,\mathrm{d}x = \int g(x)\,\mathrm{d}x = 1$. This will show $\int \chi_{[0,1]\setminus X}(x)\,\mathrm{d}x = 1 - k$.
   To prove that g is measurable we again take the sequence $X_0, X_1, X_2, \ldots$ of measurable regions and we define a sequence $u_0, u_1, u_2, \ldots$ of elementary sets of rectangles as follows. For all $n \in \mathbb{N}$ and for all $i <$ length(v_n) define $((u_n)_i)_0 = ((v_n)_i)_0$, $((u_n)_i)'_1 = \max(((v_n)_i)'_1, ((w_n)_i)'_1)$ and $((u_n)_i)''_1 = \max(((v_n)_i)''_1, ((w_n)_i)''_1)$. Now we have $\mathrm{Ar}(u_n) = \mathrm{Ar}(v_n) < \frac{1}{2^n}$.
   Furthermore, suppose $x \notin X_n$ then $x \in \mathrm{dom}(\chi_X) = \mathrm{dom}(g)$. This means either $x \in X$ or $x \in [0,1] \setminus X$.
   First suppose $x \in X$, then $\chi_X(x) = 1$ and $\chi_{[0,1]\setminus X}(x) = 0$. Suppose $x \ \varepsilon_0 \ ((u_n)_i)_0$, then $x \ \varepsilon_0 \ ((v_n)_i)_0$ and thus $1 \ \varepsilon_0 \ ((v_n)_i)_1$ and $0 \ \varepsilon_0 \ ((w_n)_i)_1$ This gives us $((v_n)_i)'_1 \leq 1 \leq ((v_n)_i)''_1$ and $((w_n)_i)'_1 \leq 0 \leq ((w_n)_i)''_1$ so $((u_n)_i)'_1 = ((v_n)_i)'_1$ and $((u_n)_i)''_1 = ((v_n)_i)''_1$ thus $g(x) \ \varepsilon_0 \ ((u_n)_i)_1$.

Suppose $x \in [0,1] \setminus X$, then $\chi_X(x) = 0$ and $\chi_{[0,1]\setminus X}(x) = 1$. Suppose $x \ \varepsilon_0 \ ((u_n)_i)_0$, then $x \ \varepsilon_0 \ ((v_n)_i)_0$ and thus $0 \ \varepsilon_0 \ ((v_n)_i)_1$ and $1 \ \varepsilon_0 \ ((w_n)_i)_1$ This gives us $((v_n)_i)'_1 \leq 0 \leq ((v_n)_i)''_1$ and $((w_n)_i)'_1 \leq 1 \leq ((w_n)_i)''_1$ so $((u_n)_i)'_1 = ((w_n)_i)'_1$ and $((u_n)_i)''_1 = ((w_n)_i)''_1$ thus $g(x) \ \varepsilon_0 \ ((w_n)_i)_1$.
What is left for us to prove is $\int \chi_X(x)\,\mathrm{d}x + \int \chi_{[0,1]\setminus X}(x)\,\mathrm{d}x = \int g(x)\,\mathrm{d}x = 1$. We have: $\int \chi_X(x)\,\mathrm{d}x = \lim_{n\to\infty} \overline{I}(v_n)$ and $\int \chi_{[0,1]\setminus X}(x)\,\mathrm{d}x = \lim\limits_{n\to\infty} \underline{I}(w_n)$, so:

$$\begin{aligned}\int \chi_X(x)\,\mathrm{d}x + \int \chi_{[0,1]\setminus X}(x)\,\mathrm{d}x &= \lim_{n\to\infty} \Big(\sum_{i=1}^{\mathrm{length}(v_n)-1} (((v_n)_i)''_0 - ((v_n)_i)'_0)((v_n)_i)''_1\Big) + \\ &\lim_{n\to\infty} \Big(\sum_{i=1}^{\mathrm{length}(v_n)-1} (((v_n)_i)''_0 - ((v_n)_i)'_0)(1-(v_n)_i)''_1\Big) \\ &= \lim_{n\to\infty} \Big(\sum_{i=1}^{\mathrm{length}(v_n)-1} (((v_n)_i)''_0 - ((v_n)_i)'_0)\Big)\end{aligned}$$

Also:

$$\begin{aligned}\int g(x)\,\mathrm{d}x = \lim_{n\to\infty} \overline{I}(u_n) &= \lim_{n\to\infty} \Big(\sum_{i=1}^{k} (((v_n)_i)''_0 - ((v_n)_i)'_0)((u_n)_i)''_1\Big) \\ &= \lim_{n\to\infty} \Big(\sum_{i=1}^{k} (((v_n)_i)''_0 - ((v_n)_i)'_0) \max(((v_n)_i)''_1, ((w_n)_i)''_1)\Big) \\ &= \lim_{n\to\infty} \Big(\sum_{i=1}^{k} (((v_n)_i)''_0 - ((v_n)_i)'_0)\Big)\end{aligned}$$

Define $f : [0,1] \to \mathbb{R}$ with $f(x) = 1$ for all $x \in [0,1]$. By lemma 3.3.10 $\mathrm{dom}(g)$ is almost full so $f(x) = g(x)$ almost everywhere, thus by lemma 3.3.3 $\int g(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x = 1$. □

4 Geometric types

The examples of possible pseudofull subsets that Brouwer gives to define discontinuous functions fall into certain equivalence classes that Brouwer calls geometric types. In this chapter we define what a geometric type is and discuss some intuitionistic mathematics on geometric types.

Definition 4.1. *Two sets $V, W \subseteq [0,1]$ are of the same* ***geometric type*** *if there exists a uniformly continuous bijection $f : [0,1] \to [0,1]$ such that $f(V) = W$ and such that its inverse f^{-1} is uniformly continuous as well. Notation: $V \sim W$.*

We now show that the properties of sets defined in definition 1.5 may be considered as properties of geometric types.

Lemma 4.2. *Suppose $V, W \subseteq [0,1]$ and suppose $V \sim W$. Then:*

(i) *If $V \equiv [0,1]$ then $W \equiv [0,1]$*

(ii) *If $V \not\equiv [0,1]$ then $W \not\equiv [0,1]$*

(iii) *If $V \# [0,1]$ then $W \# [0,1]$*

(iv) *If $\neg[V \equiv [0,1]]$ then $\neg[W \equiv [0,1]]$*

(v) *If $\neg\neg[V \equiv [0,1]]$ then $\neg\neg[W \equiv [0,1]]$*

(vi) *If $\neg[V \not\equiv [0,1]]$ then $\neg[W \not\equiv [0,1]]$*

(vii) *If $\neg\neg[V \not\equiv [0,1]]$ then $\neg\neg[W \not\equiv [0,1]]$*

(viii) *If $\neg[V \# [0,1]]$ then $\neg[W \# [0,1]]$*

(ix) *If $\neg\neg[V \# [0,1]]$ then $\neg\neg[W \# [0,1]]$*

Proof. We will prove $(i), (ii)$ and (iii). The others will then follow directly. Suppose $V, W \subseteq [0,1]$ and suppose $V \sim W$. Then there exists an uniformly continuous function bijection $f : V \to W$ such that $f(V) = W$ and such that it's inverse f^{-1} is also uniformly continuous.

(i) We have to show $\forall x \in [0,1] \exists w \in W\ [w \equiv x]$ and $\forall w \in W \exists x \in [0,1]\ [x \equiv w]$. Since $W \subseteq [0,1]$ obviously $\forall w \in W \exists x \in [0,1]\ [x \equiv w]$. Now, pick $x \in [0,1]$. Consider $f(x)$. $V \equiv [0,1]$, so find $v \in V$ such that $f(x) \equiv v$. Define $w = f^{-1}(v) \in W$. Since $f(x) \equiv v$ and since f^{-1} is a function we know $x \equiv w$.

(ii) We have to show $\exists x \in [0,1]\ \neg\big[\exists w \in W\ [x \equiv w]\big]$ or $\exists w \in W\ \neg\big[\exists x \in [0,1]\ [x \equiv w]\big]$. Since $W \subseteq [0,1]$, obviously $\exists w \in W\ \neg\big[\exists x \in [0,1]\ [x \equiv w]\big]$ can not be true. Now suppose $\neg[V \equiv [0,1]]$ then, also since $V \subseteq [0,1]$, we must have $\exists y \in [0,1]\ [\neg[\exists v \in V\ [y \equiv v]]$. Find this y and consider $x = f(y)$. Now suppose $\exists w \in W$ s.t. $x \equiv w$. Then, since f^{-1} is a function, we have $y \equiv f^{-1}(w)$. Since $f^{-1}(w) \in V$, this is a contradiction, so $\neg[\exists w \in W$ s.t. $x \equiv w]$.

(iii) We have to show $\exists x \in [0,1] \forall w \in W[x \# w]$ or $\exists w \in W \forall x \in [0,1][x \# w]$. Since $W \subseteq [0,1]$, obviously $\exists w \in W \forall x \in [0,1][x \# w]$ can not be true. Now suppose $[V \# [0,1]]$ then, also since $V \subseteq [0,1]$, we must have $\exists y \in [0,1] \forall v \in V[y \# v]$. Find this y and consider $x = f(y)$. Now pick any $w \in W$. We know, since $f^{-1}(w) \in V$, $y \# f^{-1}(w)$ and so $f^{-1}(x) \# f^{-1}(w)$. By lemma 2.1.12 we have $x \# w$.

□

The following lemma follows from the fact that the composition of two continuous functions is continuous.

Lemma 4.3. *Suppose $V, W \subseteq [0,1]$ and suppose $V \sim W$. If every function $h : V \to \mathbb{R}$ is continuous then every function $f : W \to \mathbb{R}$ is continuous.*

Proof. Suppose $f : W \to \mathbb{R}$ is a function. Pick $x \in W$ and $m \in \mathbb{N}$. We want to find $n \in \mathbb{N}$ such that for all $y \in W$ if $|x-y| < \frac{1}{n}$ then $|f(x) - f(y)| < \frac{1}{m}$. Find a uniformly continuous bijection $g : [0,1] \to [0,1]$ such that $g(V) = W$ and such that its inverse g^{-1} is also uniformly continuous. Consider $h : V \to \mathbb{R}$ with $h(v) = f(g(v))$ for all $v \in V$. This function is continuous, since every function from V to $\mathbb{R}$ is, so for every $z \in V$ there exists $k \in \mathbb{N}$ such that for all $w \in V$ if $|z - w| < \frac{1}{k}$ then $|h(z) - h(w)| < \frac{1}{m}$. Find $n \in \mathbb{N}$ such that for all $x, y \in [0,1]$ if $|x - y| < \frac{1}{n}$ then $|g^{-1}(x) - g^{-1}(y)| < \frac{1}{k}$. We claim this is the n we are looking for. Pick any $y \in W$ and find $z, w \in V$ such that $g(z) = x$ and $g(w) = y$. Find $k \in \mathbb{N}$ such that for all $w \in V$ if $|z-w| < \frac{1}{k}$ then $|h(z) - h(w)| < \frac{1}{m}$. Suppose $|x - y| < \frac{1}{n}$ then $|g(z) - g(w)| < \frac{1}{n}$ so $|g^{-1}(g(z)) - g^{-1}(g(w))| < \frac{1}{k}$ so $|z - w| < \frac{1}{k}$ so $|h(z) - h(w)| < \frac{1}{m}$ and thus $|f(g(z)) - f(g(w))| < \frac{1}{m}$ so $|f(x) - f(y)| < \frac{1}{m}$. □

Corollary 4.4. *Suppose $V, W \subseteq [0,1]$ and suppose $V \sim W$. If there exists a function $f : V \to \mathbb{R}$ such that f is discontinuous then there exists a function $h : W \to \mathbb{R}$ such that h is discontinuous.*

Proof. Find $f : V \to \mathbb{R}$ such that f is discontinuous. Find $x \in V$ and $n \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ exists $y \in V$ with $|x - y| \leq \frac{1}{2^m}$ but $|f(x) - f(y)| > \frac{1}{2^n}$. Also, find a uniformly continuous bijection $g : [0,1] \to [0,1]$ such that $g(W) = V$ and such that its inverse g^{-1} is also uniformly continuous. We define $h : W \to \mathbb{R}$ with $h(v) = f(g(v))$ for all $v \in W$. We will prove h is discontinuous. For this, consider $z := g^{-1}(x) \in W$ and n. Pick any $m \in \mathbb{N}$. Find $k \in \mathbb{N}$ such that for all $y \in \mathbb{R}$ if $|x - y| \leq \frac{1}{2^k}$ then $|g^{-1}(x) - g^{-1}(y)| \leq \frac{1}{2^m}$. Now, find $y \in V$ such that $|x - y| \leq \frac{1}{2^k}$ but $|f(x) - f(y)| > \frac{1}{2^n}$. We claim $w := g^{-1}(y) \in W$ is such that $|z - w| \leq \frac{1}{2^m}$ but $|h(z) - h(w)| > \frac{1}{2^n}$. Since $|x - y| \leq \frac{1}{2^k}$ we have $|g^{-1}(x) - g^{-1}(y)| \leq \frac{1}{2^m}$ so $|z - w| \leq \frac{1}{2^m}$. Also $|f(x) - f(y)| > \frac{1}{2^n}$ so $|f(g(g^{-1}(x))) - f(g(g^{-1}(y)))| \leq \frac{1}{2^n}$ so $|h(z) - h(w)| > \frac{1}{2^n}$. So h is discontinuous. □

For the next lemma we need a couple of definitions.

Definition 4.5. *X is a **totally bounded** (Brouwer: katalogisiert) set if for all $m \in \mathbb{N}$ there exists $p_0, p_1, \ldots, p_{n-1} \in X$ such that for all $q \in X$ there exists $i < n$ with $|q - p_i| < \frac{1}{m}$.*

Definition 4.6. *A point $x \in [0,1]$ is a **closure point** of X if for every $n \in \mathbb{N}$ there exists a $y \in X$ such that $|x - y| < 2^{-n}$.*

Definition 4.7. *The **closure** of X is $\bar{X} := \{x \mid x \in [0,1] \mid x$ is a closurepoint of $X\}$, the set of closure points of X.*

Definition 4.8. *A set X is **closed** if $X \equiv \bar{X}$.*

Definition 4.9. *X is a **perfect** set if it is closed and if for all $x \in X$ and every $n \in \mathbb{N}$ there exists $y \in X$ such that $0 < |x - y| < \frac{1}{n}$.*

Lemma 4.10. *Suppose $V, W \subseteq [0,1]$ and suppose $V \sim W$. Also suppose there exists a totally bounded perfect set X such that for each $x \in X$ we can not prove $x \in V$. Then there exists a totally bounded perfect set Y such that for each $y \in Y$ we can not prove $y \in W$.*

Proof. Since we have $V \sim W$, there exists an uniformly continuous function bijection $f : V \to W$ such that $f(V) = W$ and such that its inverse f^{-1} is also uniformly continuous. Define $Y = f(X)$. First we will prove that Y is totally bounded, then we will prove that Y is closed, then that Y is perfect and lastly we will prove that for each $y \in Y$ we can not prove $y \in W$.
To prove that Y is totally bounded pick any $m \in \mathbb{N}$. Find $k \in \mathbb{N}$ such that for all $x, y \in [0, 1]$ if $|x - y| < \frac{1}{k}$ then $|f(x) - f(y)| < \frac{1}{m}$. For this k, find $p_0, p_1, \ldots, p_{n-1} \in X$ such that for all $x \in X$ there exists $i < n$ with $|q - p_i| < \frac{1}{k}$. Consider the sequence $p'_0 = f(p_0), p'_1 = f(p_1), \ldots, p'_{n-1} = f(p_{n-1})$. Pick any $y \in Y$ and define $x = f^{-1}(y)$. Then $x \in X$ so find $i < n$ such that $|x - p_i| < \frac{1}{k}$. But this means $|y - p'_i| < \frac{1}{m}$.
To show that Y is closed pick any $y \in [0, 1]$ which is a closure point of Y. This means for all $n \in \mathbb{N}$ there exists $y_n \in Y$ such that $|y_n - y| < 2^{-n}$. Since f^{-1} is uniformly continuous we know for all $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that if $|y - x| < \frac{1}{k}$ then $|f^{-1}(y) - f^{-1}(x)| < \frac{1}{m}$. Fix m and find k such that $|y - x| < \frac{1}{k}$ then $|f^{-1}(y) - f^{-1}(x)| < \frac{1}{m}$. Pick n such that $2^{-n} \leq \frac{1}{k}$ and find y_n such that $|y_n - y| \leq 2^{-n} \leq \frac{1}{k}$. Then $|f^{-1}(y_n) - f^{-1}(y)| \leq \frac{1}{m}$. So $f^{-1}(y)$ is a closure point of X and thus $f^{-1}(x) \in X$, so $y \in Y$.
Now we will show that Y is perfect. Pick any $y \in Y$ and $n \in \mathbb{N}$. Now find $m \in \mathbb{N}$ such that for all $x, y \in [0, 1]$ if $|x - y| < \frac{1}{m}$ then $|f(x) - f(y)| < \frac{1}{n}$. Define $x = f^{-1}(y)$, then $x \in X$ and thus there exists x' such that $0 < |x - x'| < \frac{1}{m}$. Now $0 < |y - f(x')| < \frac{1}{n}$ and $f(x') \in Y$.
Lastly we will show that for each $y \in Y$ we can not prove $y \in W$. Suppose $y \in Y$ and suppose $y \in W$. Consider $f^{-1}(y)$. $f^{-1}(y) \in X$, but since $y \in W$, $f^{-1}(y) \in V$, which we can not prove. □

Lemma 4.11. *Suppose $V, W \subseteq [0, 1]$ and suppose $V \sim W$. If there exists a sequence $v_1, v_2, v_3, \dots \in [0, 1]$ such that:*

(i) For every $i \neq j$, $v_i \neq v_j$, and

(ii) for every $i \in \mathbb{N}$ we can not prove $v_i \in V$

then there exists a sequence $w_1, w_2, w_3, \dots \in [0, 1]$ such that (i) and (ii) hold for W.

Proof. Find a uniformly continuous bijection $f : [0, 1] \to [0, 1]$ such that $f(V) = W$ and such that its inverse f^{-1} is also uniformly continuous. Suppose there exists a sequence $v_1, v_2, v_3, \dots \in [0, 1]$ such that (i) and (ii) hold. Consider $w_1 = f(v_1), w_2 = f(v_2), w_3 = f(v_3)$. Suppose $i \neq j \in \mathbb{N}$, then $v_i \neq v_j$ so $f(v_i) \neq f(v_j)$ since f is a bijection. Also, suppose we can find $i \in \mathbb{N}$ such that we can prove $w_i \in W$. Then $f^{-1}(w_i) = f^{-1}(f(v_i)) = v_i \in V$, which is a contradiction. So for every $i \in \mathbb{N}$ we can not prove $w_i \in W$. □

Lemma 4.12. *Suppose $V, W \subseteq [0, 1]$ and suppose $V \sim W$. Suppose there exists a set $X \subset [0, 1]$ which is dense in $[0, 1]$ and such that for all $x \in X$ we can not prove $x \in V$ and such that for all $x, y \in X$ we have $x = y$ or $x \# y$. Then there exists a set $Y \subset [0, 1]$ which is dense in $[0, 1]$ and such that for all $y \in Y$ we can not prove $y \in W$ and such that for all $x, y \in Y$ we have $x = y$ or $x \# y$.*

Proof. Find a uniformly continuous bijection $f : [0, 1] \to [0, 1]$ such that $f(V) = W$ and such that its inverse f^{-1} is also uniformly continuous. Suppose $X \subset [0, 1]$ is dense in $[0, 1]$ and such that for all $x \in X$ we can not prove $x \in V$ and such that for all $x, y \in X$ we have $x = y$ or $x \# y$. Define $Y := f(X)$. Suppose $y \in Y$ and suppose we can prove $y \in W$ then $f^{-1}(y) \in X$ and $f^{-1}(y) \in V$, which is a contradiction. Thus we can not prove $y \in W$. Suppose $x, y \in Y$. Then $f^{-1}(x), f^{-1}(y) \in X$ thus $f^{-1}(x) = f^{-1}(y)$ or $f^{-1}(x) \# f^{-1}(y)$. Suppose $f^{-1}(x) = f^{-1}(y)$ then $x = y$. Suppose $f^{-1}(x) \# f^{-1}(y)$, then since f^{-1} is continuous $x \# y$. Also, since f and f^{-1} are continuous and bijections, by lemma 2.2.10 f and f^{-1} are monotone. Pick $x, y \in [0, 1]$

such that $x < y$, then $f^{-1}(x) < f^{-1}(y)$. Since X is dense in $[0, 1]$ there exists $z \in X$ such that $f^{-1}(x) < z < f^{-1}(y)$. Thus $x < f(z) < y$ and $f(z) \in Y$, so Y is dense in $[0, 1]$. □

5 Examples

In this chapter we will take a look at the examples Brouwer gives.

5.1 Example 1

Brouwer defines A to be the geometric type of all the sets to which a real number belongs if and only if the law of the excluded middle is true. It is unclear here what Brouwer really means. There are two possible interpretations for this geometric type, namely:

(i) A is the geometric type of H with $H = \{x \mid x \in [0,1] \mid \forall \text{ propositions } P\ [P \vee \neg P]\}$

(ii) Every proposition P defines a geometric type A_P. Suppose P is a proposition, then A_P is the geometric type of $H_P = \{x \mid x \in [0,1] \mid P \vee \neg P\}$

We will consider these two options a bit more closely.
Suppose we would use option (i). For all $x \in [0,1]$ we have $\neg x \in H$, since, by lemma 2.2.8, we know $\neg \forall x \in [0,1]\ [(x = 0) \vee \neg(x = 0)]$. This gives us $H = \emptyset$. We can also consider the complement of H. We then get $[0,1] \setminus H = \{x \mid x \in [0,1] \mid \neg\forall \text{ propositions } P\ [P \vee \neg P]\} = [0,1]$. This shows us that Brouwer did not mean option (i), since he claims that the complement of the set that he proposes can not contain any real number.
So we will assume he means option (ii) and work with this option from now on. Suppose for P we take the proposition that π contains a block of nine consecutive nines in its decimal expansion. So $P = \exists n \in \mathbb{N}[n = k_1]$ as in definition 1.3.1. This means there are propositions P for which we can not prove $\exists x \in H_P$. Again, for every proposition P we can consider the complement of H_P:

$$\begin{aligned}[0,1] \setminus H_P &= \{x \mid x \in [0,1] \mid \neg(P \vee \neg P)\} \\ &= \{\ x \mid x \in [0,1] \mid \neg P \wedge \neg\neg P)\} \\ &= \emptyset.\end{aligned}$$

Again, take $P = \exists n \in \mathbb{N}[n = k_1]$. Now suppose H_P is measurable, then find $k \in \mathbb{N}$ such that $\mu(H_P) = k$. This means also $[0,1] \setminus H_P$ is measurable and $\mu([0,1] \setminus H_P) = 1 - k$. But $[0,1] \setminus H_P = \emptyset$, so $1 - k = 0$. Thus $k = 1$, which means H_P is almost full. By lemma 3.2.2 this means there exists $x \in H_P$, but we can not prove this. Thus there does not exists a measure of H_P. So, there are propositions P for which we can not prove that H_P has a measure.

Lemma 5.1.1. *For every proposition P, every representative of A_P seems to coincide with $[0,1]$.*

Proof. Suppose P is a proposition. By lemma 4.2 it is sufficient to prove $\neg\neg\big[H_P \equiv [0,1]\big]$. Suppose $P \vee \neg P$ then $H_P = [0,1]$, so $H_P \equiv [0,1]$. So if $\neg[H_P \equiv [0,1]]$ then $\neg(P \vee \neg P)$. Thus if $\neg\neg(P \vee \neg P)$ then $\neg\neg[H_P \equiv [0,1]]$. And since for every proposition Q, $\neg\neg(Q \vee \neg Q)$, we have $\neg\neg(P \vee \neg P)$ so H_P seems to coincide with $[0,1]$. □

5.2 Example 2

We define B_1 to be the geometric type of I_1 [(2)] and B_2 to be the geometric type of I_2 [(3)]:

$$\begin{aligned} I_1 &= \{\ x \mid x \in [0,1] \mid x = \sum_{n=1}^{\infty} \frac{a_n}{2^n} \mid \forall k \in \mathbb{N}\ \exists m > k, n_1 > m, n_2 > m[a_m = a_{n_1} = 1, a_{n_2} = 0]\ \} \\ &= \{\ x \mid x \in [0,1] \mid x = \sum_{n=1}^{\infty} \frac{a_n}{2^n} \mid \forall m \in \mathbb{N}\ \exists n \in \mathbb{N}\ [n > m\ \wedge\ a_m \neq a_n]\ \} \end{aligned}$$

$$\begin{aligned} I_2 &= \{\ x \mid x \in [0,1] \mid \forall q \in \mathbb{Q}\ \exists m \in \mathbb{N}\ |x - q| > 1/m\} \\ &= \{\ x \mid x \in [0,1] \mid \forall q \in \mathbb{Q}\ [q \# x]\} \end{aligned}$$

Lemma 5.2.1. *$I_1 = \big\{\ x \mid x \in [0,1] \mid \forall q \in \mathbb{Q}'\ \exists m \in \mathbb{N}\ |x - q| > 1/m\}$ where $\mathbb{Q}' = \{\ q \in \mathbb{Q} \mid \exists m \in \mathbb{N}[q = \sum_{n=1}^{m} \frac{a_n}{2^n}$ with $a_n \in \{0,1\}] \vee q = 1\big\}$.*

Proof. Suppose $x \in I_1$. Then $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$ s.t. $\forall m \in \mathbb{N}\ \exists n \in \mathbb{N}$ with $n > m$ and $a_m \neq a_n$. Pick any $q \in \mathbb{Q}'$. This means $q = \sum_{n=1}^{m} \frac{b_n}{2^n}$ for some $m \in \mathbb{N}$ and with $b_n \in \{0,1\}$. Consider $x' = \sum_{n=1}^{m} \frac{a_n}{2^n}$.

- Suppose $x' > q$, then find $k \in \mathbb{N}$ such that $|x' - q| > \frac{1}{k} \geq \frac{1}{2^m}$. Since $x \geq x'$ we have $|x - q| \geq |x' - q| > \frac{1}{k} \geq \frac{1}{2^m}$.
- Suppose $x' = q$. Since for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $n > m$ and $a_m \neq a_n$ we can find $k > m \in \mathbb{N}$ such that $a_k = 1$. This means $|x - q| > \frac{1}{2^{k+1}}$.
- Suppose $x' < q$. We know $\sum_{n=m+1}^{\infty} \frac{a_n}{2^n} < \sum_{n=m+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^m}$, since we can find $k > m$ such that $a_k = 0$. So $|x - x'| = \sum_{n=m+1}^{\infty} \frac{a_n}{2^n} \leq (\frac{1}{2^m} - \frac{1}{2^k+1})$. Also we know $|x' - x| + |x - q| \geq |x' - q| \geq \frac{1}{2^m}$. So we get $|x - q| \geq (\frac{1}{2^m} - \sum_{n=m+1}^{\infty} \frac{a_n}{2^n}) > \frac{1}{2^m} - (\frac{1}{2^m} - \frac{1}{2^k+1}) = \frac{1}{2^k+1}$.

Now suppose $x \in [0,1]$ and $\forall q \in \mathbb{Q}' \exists m \in \mathbb{N} |x - q| > \frac{1}{m}$. Because $\forall q \in \mathbb{Q}' \exists m \in \mathbb{N} |x - q| > \frac{1}{m}$ we can find a sequence $(a_n)_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$. We have to prove $\forall m \in \mathbb{N}\ \exists p \in \mathbb{N}$ with $p > m$ and $a_m \neq a_p$. Suppose $m \in \mathbb{N}$. Define $x' = \sum_{n=1}^{m} \frac{a_n}{2^n}$.

- Suppose $a_m = 0$. $x' \in \mathbb{Q}'$, so find $k \in \mathbb{N}$ such that $|x' - x| = \sum_{n=m+1}^{\infty} \frac{a_n}{2^n} > \frac{1}{2^k} > 0$. So there exists $p > m \in \mathbb{N}$ such that $a_p = 1$.
- Suppose $a_m = 1$. Define $x'' = x' + \sum_{n=m+1}^{\infty} \frac{1}{2^n}$.
  If there exists $n \leq m$ such that $a_n = 0$, define $l = \max\{n \leq m | a_n = 0\}$. So $a_n = 1$ for all $n > l$, which means $x'' = \sum_{n=1}^{l-1} \frac{a_n}{2^n} + \frac{1}{2^l}$, so $x'' \in \mathbb{Q}'$. So find $k \in \mathbb{N}$ such that $|x - x''| = |\frac{1}{2^l} - \sum_{n=l}^{\infty} \frac{a_n}{2^n}| > \frac{1}{k} > 0$. So there exists $p \geq l$ such that $a_p = 0$. But since l was the biggest such that $l \leq m$ and such that $a_l = 0$ we know $p > m$.
  If $a_n = 1$ for all $n \leq m$, then $x'' = 1$, so $x'' \in \mathbb{Q}'$. Find $k \in \mathbb{N}$ such $|1 - x| \geq \frac{1}{k} > 0$. So there exists $p \in \mathbb{N}$ such that $a_p = 0$. But since $a_n = 1$ for all $n \leq m$ we know $p > m$.

□

Even though Brouwer seems to suggest that B_1 and B_2 are of a different geometric type, we claim they are the same geometric type. The following two lemmas prove this.

(2) This is G_1 from example 2 of Brouwers article.

(3) In example 2 of Brouwers article this is called "der Spezies der positiv-irrationalen".

Lemma 5.2.2. *Suppose $(c_n)_{n=1}^{\infty}$ and $(d_n)_{n=1}^{\infty}$ are countable sequences with:*

- $\forall n \in \mathbb{N}[c_n, d_n \in [0,1]]$
- $\forall n \neq m \in \mathbb{N}[c_n \# c_m \wedge d_n \# d_m]$
- *The sequences $(c_n)_{n=1}^{\infty}$ and $(d_n)_{n=1}^{\infty}$ are dense in $[0,1]$.*
- $\exists n, m, k, l \in \mathbb{N}[c_n = d_k = 0 \wedge c_m = d_l = 1$

Also suppose $g : (c_n)_{n=1}^{\infty} \to (d_n)_{n=1}^{\infty}$ is an isomorphism, that is g is bijection and for all $n, m \in \mathbb{N}$ if $c_n < c_m$ then $g(c_n) < g(c_m)$. Then g is uniformly continuous.

Proof. Pick $m \in \mathbb{N}$. Split the interval $[0,1]$ in smaller intervals $[d_{n_0}, d_{n_1}], [d_{n_1}, d_{n_2}], \ldots, [d_{n_{k-1}}, d_{n_k}]$ with $d_{n_0} = 0, d_{n_k} = 1$, $n_i \neq n_j$ for all $i, j \leq k$ and such that for all $i < k$, $|d_{n_i} - d_{n_{i+1}}| \leq 2^{-m-1}$. This is possible since $(d_n)_{n=1}^{\infty}$ is dense in $[0,1]$. Now consider $g^{-1}(d_{n_0}), \ldots, g^{-1}(d_{n_k})$. Since g is order preserving we must have $g^{-1}(d_{n_0}) = 0$ and $g^{-1}(d_{n_k}) = 1$. Define $\delta := \min\{|g^{-1}(d_{n_i}) - g^{-1}(d_{n_{i+1}})| \bigm| i < k\}$ and find $n \in \mathbb{N}$ such that $2^{-n} \leq \delta$. We claim: for all $p, q \in \mathbb{N}$ if $|c_p - c_q| \leq 2^{-n}$ then $|g(c_p) - g(c_q)| \leq 2^{-m}$. Now pick $q, p \in \mathbb{N}$ and suppose $|c_p - c_q| \leq 2^{-n}$. Then there exists $i < k-1$ such that $g^{-1}(d_{n_i}) \leq c_p, c_q \leq g^{-1}(d_{n_{i+2}})$. This gives, since g is order preserving, $d_{n_i} \leq g(c_p), g_(c_q) \leq d_{n_{i+2}}$ so $|g(c_p) - g(c_q)| \leq 2^{-m-1} + 2^{-m-1} = 2^{-m}$. □

Lemma 5.2.3. *Suppose $c = (c_n)_{n=1}^{\infty}$ and $d = (d_n)_{n=1}^{\infty}$ are countable sequences with:*

- $\forall n \in \mathbb{N}[c_n, d_n \in [0,1]]$
- $\forall n \neq m \in \mathbb{N}[c_n \# c_m \wedge d_n \# d_m]$
- *The sequences $(c_n)_{n=1}^{\infty}$ and $(d_n)_{n=1}^{\infty}$ are dense in $[0,1]$*
- $\exists n, m, k, l \in \mathbb{N}[c_n = d_k = 0 \wedge c_m = d_l = 1$

Define $I_c = \{\ x \mid x \in [0,1] \mid \forall n \in \mathbb{N}\ [x \# c_n]\}$ and $I_d = \{\ x \mid x \in [0,1] \mid \forall n \in \mathbb{N}\ [x \# d_n]\}$. Then $I_c \sim I_d$ and also $I_c \cup \{c_i | i \in \mathbb{N}\} \sim I_d \cup \{d_i | i \in \mathbb{N}\}$.

Proof. We use the back-and-forth method to find an isomorphism $g : (c_n)_{n=1}^{\infty} \to (d_n)_{n=1}^{\infty}$. With this isomorphism we will define a bijection $f : [0,1] \to [0,1]$, such that $f(I_c) = I_d$. Suppose $x \in [0,1]$. Consider $x(0), x(1), x(2), \ldots$. We will define $f(x) = (f(x))(0), (f(x))(1), (f(x))(2), \ldots$ with induction. First we define $(f(x))'(0)$ and $(f(x))''(0)$. Suppose $x'(0) < 0$. Define $c_{x'(0)} = x'(0)$ and $(f(x))'(0) = x'(0)$. Suppose $x'(0) \geq 0$, then find $n \in \mathbb{N}$ such that $c_n \leq x'(0)$. Define $c_{x'(0)} := c_n$ and $(f(x))'(0) = g(c_{x'(0)})$. Now suppose $x''(0) > 1$ then define $c_{x''(0)} = x''(0)$ and $(f(x))''(0) = x''(0)$. Suppose $x''(0) \leq 1$ then find $m \in \mathbb{N}$ such that $x''(0) \leq c_m$. Define $c_{x''(0)} := c_m$ and $(f(x))''(0) = g(c_{x''(0)})$. Now suppose $(f(x))(0), (f(x))(1), \ldots, (f(x))(n)$ are defined. Again, suppose $x'(n+1) < 0$ then define $c_{x'(n+1)} := x'(n+1)$ and $(f(x))'(n+1) = x'(n+1)$. Suppose $x'(n+1) \geq 0$ then find $s \in \mathbb{N}$ such that $x'(n) < c_s \leq x'(n+1)$. Define $c_{x'(n+1)} := c_s$ and $(f(x))'(n+1) = g(c_{x'(n+1)})$. Also, suppose $x''(n+1) > 1$ then define $c_{x''(n+1)} = x''(n+1)$ and $(f(x))''(n+1) = x''(n+1)$. Suppose $x''(n+1) \leq 1$ then find $t \in \mathbb{N}$ such that $x''(n+1) \leq c_t < x''(n)$. Define $c_{x''(n+1)} := c_t$ and $(f(x))''(n+1) = g(c_{x''(n+1)})$. Proving that $(f(x))(0), (f(x))(1), (f(x))(2), \ldots$ is a real number is very straightforward. Furthermore we have to show the following:

(i) For every $x \# y \in [0,1]$ we have $f(x) \# f(y)$.

(ii) For every $y \in [0,1]$ there exists $x \in [0,1]$ such that $f(x) = y$.

(iii) $f(I_c) = I_d$

We will now prove properties (i), (ii) and (iii).

(i) Pick $x\#y \in [0,1]$, then there exists $n, m \in \mathbb{N}$ such that $x'(n) > y''(m)$ or $x''(n) < y'(m)$. Suppose $x'(n) > y''(m)$. The other case is similar. $x'(n) > 0$ since $y''(m) > 0$ so $(f(x))'(n) = g(c_{x'(n)})$. Also $y''(m) < 1$ since $x'(n) < 1$ so $(f(y))''(m) = g(c_{y''(m)})$. Furthermore $c_{x(n+1)'} > x'(n) > y''(m) > c_{y(m+1)''}$ so $g(c_{x(n+1)'}) > g(c_{y(m+1)''})$ and thus $(f(x))(n+1)' > (f(y))(m+1)''$.

(ii) Suppose $y \in [0,1]$, then $y \equiv d_y$ where $d_y := (d_{y'(0)}, d_{y''(0)}), (d_{y'(1)}, d_{y''(1)}), \ldots$. Now consider $z := z(0), z(1), z(2), \ldots$ where, for every $i \in \mathbb{N}$, $z'(i) = d_{y'(i)}$ if $d_{y'(i)} < 0$ and $z'(i) = g^{-1}(d_{y'(i)})$ else, and $z''(i) = d_{y''(i)}$ if $d_{y''(i)} > 1$ and $g^{-1}(d_{y''(i)})$ else. Then $f(z) \equiv y$.

(iii) Suppose $x \in I_c$ then for every $n \in \mathbb{N}$, $x \# c_n$. Pick $n \in \mathbb{N}$. We want to show there exists $k, p \in \mathbb{N}$ such that $(f(x))'(p) > d''_n(k)$ or $(f(x))''(p) < d'_n(k)$. Since $x \# g^{-1}(d_n)$ for every $n \in \mathbb{N}$ we have there exists $m \in \mathbb{N}$ such that $x'(m) > g^{-1}(d_n)$ or $x''(m) < g^{-1}(d_n)$. Find this m, k and suppose $x(m)' > g^{-1}(d_n)$. The other case is similar. Since $x(m)' > g^{-1}(d_n)$ we have $x'(m) > 0$ so $(f(x))'(m) = g(c_{x'(m)})$. We have $g^{-1}(d_n) < x(m)' < c_{x(m+1)'} < x(m+1)' < x(m+1)'' < c_{x(m+1)''} < x(m)''$ and thus $d_n < g(c_{x(m+1)'})$ so $d''_n < (f(x))(m+1)'$. So $d_n \# f(x)$, thus $f(x) \in I_d$.

This proves $I_c \sim I_d$. To prove $I_c \cup \{c_i | i \in \mathbb{N}\} \sim I_d \cup \{d_i | i \in \mathbb{N}\}$ we have to show $f(\{c_i | i \in \mathbb{N}\}) = \{d_i | i \in \mathbb{N}\}$. Suppose $x \in \{c_i | i \in \mathbb{N}\}$ and $x = x(0), x(1), x(2), \ldots$. Then $x \equiv c_x$ where $c_x := (c_{x'(0)}, c_{x''(0)}), (c_{x'(1)}, c_{x''(1)}), \ldots$. So, for every $n \in \mathbb{N}$ we have $c_{x'(n)} < x < c_{x''(n)}$ and thus for every $n \in \mathbb{N}$ we have $g(c_{x'(n)}) < g(x) < g(c_{x''(n)})$ so $f(x) \equiv g(x)$ and thus $f(x) \in \{d_i | i \in \mathbb{N}\}$. □

From now on, we will call B the geometric type of I_1 and I_2.

Lemma 5.2.4. *Every representative of B is of the form $I_c := \{x | x \in [0,1] | \forall n \in \mathbb{N}[x \# c_n] \}$ for some countable sequence $c = (c_n)_{n=0}^{\infty}$ with:*

(i) $\forall n \in \mathbb{N}[c_n \in [0,1]]$

(ii) $\forall n \neq m \in \mathbb{N}[c_n \# c_m]$

(iii) $(c_n)_{n=0}^{\infty}$ *is dense in* $[0,1]$

(iv) $\exists n, m \in \mathbb{N}[c_n = 0 \wedge c_m = 1]$

Proof. Suppose B' is a representative of B, then $I_2 \sim B'$. This means there exists a uniformly continuous bijection $f : [0,1] \to [0,1]$ such that $f(I_2) = B'$. Now find an enumeration $q_0, q_1, q_2, \ldots$ of $\mathbb{Q}$ and consider $c_0 := f(q_0), c_1 := f(q_1), c_2 := f(q_2), \ldots$. This sequence is a sequence such that conditions $(i) - (iv)$ are satisfied and such that $B' = I_c$. We will now prove this:

(i) Obviously, for all $n \in \mathbb{N}$ we have $c_n \in [0,1]$.

(ii) Since for all $n, m \in \mathbb{N}$ we have $q_n \# q_m$ and f is a bijection we have $f(q_n) \# f(q_m)$ and thus $c_n \# c_m$.

(iii) f is a continuous bijection thus, by lemma 2.2.10, f is monotone. Now suppose $y_1 < y_2 \in [0,1]$ then there exists $x_1 < x_2 \in [0,1]$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Also, there exists $q_n \in \mathbb{Q}$ such that $x_1 < q_n < x_2$, and thus $y_1 < c_n < y_2$.

(iv) Since, by lemma 2.2.10, f is monotone we must have $f(0) = 0$ and $f(1) = 1$ or $f(0) = 1$ and $f(1) = 0$.

Now we will show $f(I_2) = I_c$, which proves $B' = I_c$. Suppose $x \in I_2$ then for all $n \in \mathbb{N}$, $q_n \# x$. This means, since f is a bijection, for all $n \in \mathbb{N}$, $f(q_n) \# f(x)$ and thus $f(x) \in I_c$. So $f(I_2) \subseteq I_c$. Now suppose $x \in I_c$ then for all $n \in \mathbb{N}$, $c_n \# x$. This means, since f is bijection, for all $n \in \mathbb{N}$, $f^{-1}(c_n) \# f^{-1}(x)$. Thus $f^{-1}(x) \in I_2$ and thus $x \in f(I_2)$. So $I_c \subseteq f(I_2)$. □

Lemma 5.2.5. *Every representative of B has measure* 1.

Proof. Suppose B' is a representative of B. By lemma 5.2.4 we know it is of the form $I_c := \{x | x \in [0,1] | \forall n \in \mathbb{N}[x \ \# \ c_n] \ \}$ for some countable sequence $c = (c_n)_{n=0}^{\infty}$ satisfying condition $(i), (ii), (iii)$ and (iv) of 5.2.4. Pick any $n \in \mathbb{N}$. Define X_n to be the measurable region $\mathcal{R}(\alpha_n(0), \alpha_n(1), \alpha_n(2))$ where, for all $i \in \mathbb{N}$, $\alpha_n(i) = (c_i - \frac{1}{2^{n+2+i}}, c_i + \frac{1}{2^{n+2+i}})$. Then $\mu(X_n) \leq \sum_{i=1}^{\infty} \frac{1}{2^{n+1+i}} \leq \frac{1}{2^n}$. Define v_n to be the elementary set of rectangles defined by $(v_n)_0, \ldots, (v_n)_n$, where $\forall i \leq n \ ((v_n)_i)_0 = (\frac{i}{n+1}, \frac{i+1}{n+1})$ and $((v_n)_i)_1 = (1 - \frac{1}{2^n}, 1)$. Then $\mathrm{Ar}^*(v_n) = \frac{n+1}{2^n(n+1)} = \frac{1}{2^n}$. Pick any $x \in [0,1]$. Suppose $x \notin X_n$, then:

$$
\begin{aligned}
&\neg \exists m \in \mathbb{N} \exists k \in \mathbb{N}[\alpha_n(m)' < x'(k) \leq x''(k) < \alpha_n(m)''] \implies \\
&\forall m \in \mathbb{N} \neg \exists k \in \mathbb{N}[\alpha_n(m)' < x'(k) \leq x''(k) < \alpha_n(m)''] \implies \\
&\forall m \in \mathbb{N} \forall k \in \mathbb{N}[\alpha_n(m)' \geq x'(k) \vee x''(k) \geq \alpha_n(m)''] \implies \\
&\forall m \in \mathbb{N} \forall k \in \mathbb{N}[c_m - \frac{1}{2^{n+2+m}} \geq x'(k) \vee x''(k) \geq c_m + \frac{1}{2^{n+2+m}}]
\end{aligned}
$$

We have to show:

(i) $x \in \mathrm{dom}(\chi_{I_c})$. For this we show $x \in I_c$.

(ii) For all $i < \mathrm{length}(v_n)$ $[x \ \varepsilon_0 \ ((v_n)_i)_0 \implies \chi_{I_c}(x) \ \varepsilon_0 \ ((v_n)_i)_1]$.

First we will prove (i). For this we pick any c_i. We have to show $x \# c_i$, so we need to find a $b \in \mathbb{N}$ such that $x'(b) > c_i$ or $x''(b) < c_i$. Take a $b \in \mathbb{N}$ such that $l(x(b)) < \frac{1}{2^{n+2+i}}$. By the above, we know $x'(b) \leq c_i - \frac{1}{2^{n+2+i}}$ or $x''(b) \geq c_i + \frac{1}{2^{n+2+i}}$. First suppose $x'(b) \leq c_i - \frac{1}{2^{n+2+i}}$, this means $x''(b) < c_i$. Now suppose $x''(b) \geq c_i + \frac{1}{2^{n+2+i}}$, that means $x'(b) > c_i$.
Now we will prove (ii). For this we pick any $i < \mathrm{length}(v_n)$. We know, by the above, $x \in I_c$. Thus $\chi_{I_c}(x) = 1$. Furthermore $1 \leq 1$ and $1 - \frac{1}{n} \leq 1$ and thus $\chi_{I_c}(x) \ \varepsilon_0 \ ((v_n)_i)_1$.

So I_c is measurable and the measure of I_c is $\lim_{n\to\infty} \overline{I}(v_n) = \lim_{n\to\infty} \sum_{i=0}^{\mathrm{length}(v_n)-1} (\frac{n+1}{n+1}) = 1$. □

Lemma 5.2.6. *Every representative of B is apart from* $[0,1]$.

Proof. By lemma 4.2 it is enough to show $I_1 \# [0,1]$ to prove our claim. But this is trivial, since for example $\frac{1}{2} \# I_1$. □

Lemma 5.2.7. *For every representative B' of B we have that functions, $f : B' \to \mathbb{R}$ are continuous, but not necessarily uniformly continuous.*

To prove the above lemma we will remember the spread σ_{ir} from definition 2.1.6.

Lemma 5.2.8. *We have the following properties for σ_{ir}:*

(i) For every real number $x \in I_2$ there exists an $\alpha \in \sigma_{ir}$ such that $\alpha \equiv x$, and

(ii) for every $\alpha \in \sigma$, $\alpha \in I_2$.

Proof. First we will show (ii). Suppose $\alpha \in \sigma_{\text{ir}}$, then $\alpha \in \sigma$, so α is a real number. Furthermore, pick $n \in \mathbb{N}$, then $(q_n < \alpha'(n) \vee q_n > \alpha''(n))$. Suppose, without loss of generality, $(q_n < \alpha'(n)$. Then $|\alpha - q_n| > \alpha'(n) - q_n > 0$.
Now we will show (i). Take any real number $x \in I_2$. Since x is real number and by lemma 2.1.5 we find $\beta \in \sigma_{\text{reg}}$ such that $\beta \equiv x$. We will then find our needed α by deleting some rational segments of β. We know, since $x \in I_2$, for every $n \in \mathbb{N}$ there exist $m, k \in \mathbb{N}$ such that $x'(k) - q_n > \frac{1}{m}$ or $q_n - x''(k) > \frac{1}{m}$. Since for every $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that $\beta(l) \sqsubseteq x(k)$ we know for every $n \in \mathbb{N}$ there exists an $k \in \mathbb{N}$ such that $q_n < \beta'(k) \vee q_n > \beta''(k)$. Now define $\alpha(0)$ to be the first $\beta(m) \in \mathbb{N}$ such that $q_0 < \beta'(m) \vee q_0 > \beta''(m)$. Suppose we defined $\alpha(0), \alpha(1), \ldots, \alpha(n-1)$ then define $\alpha(n)$ to be the first $\beta(m)$ such that $\beta(m) \sqsubset \alpha(n-1)$ and such that $q_n < \beta'(m) \vee q_n > \beta''(m)$. Now also $\alpha \equiv x$ and $\alpha \in \sigma_{\text{ir}}$. □

Now we will prove lemma 5.2.7

Proof. By lemma 4.3 it is enough to show this for I_2. Thus, we will show that functions $f : I_2 \to \mathbb{R}$ are continuous. We will do this in a similar way as in the proof of theorem 2.1.11. So suppose $f : I_2 \to \mathbb{R}$ is a function. We define a function $f' : \sigma_{\text{ir}} \to \sigma_{\text{reg}}$ such that for every $\alpha \in \sigma_{\text{ir}}$, $f(\alpha) \equiv f'(\alpha)$. For every $\alpha \in \sigma_{\text{ir}}$, define $f'(\alpha) := F_{\text{reg}}(f(\alpha))$. With f' we will prove that f is continuous. Suppose $x \in I_2$ and $m \in \mathbb{N}$. We want to find $n \in \mathbb{N}$ such that for every $y \in I_2$ if $|x - y| < \frac{1}{n}$ then $|f(x) - f(y)| < \frac{1}{m}$.
Find $\alpha \in \sigma_{\text{ir}}$ such that $\alpha \equiv x$. Notice that for every $\alpha \in \sigma_{\text{ir}}$ there exists $k \in \mathbb{N}$ such that $f'(\alpha)(m+1) = k$. Thus we can find a $p \in \mathbb{N}$ such that for every $\beta \in \sigma_{\text{ir}}$ if $\bar{\beta}p = \bar{\alpha}p$ then $f'(\beta)(m+1) = f'(\alpha)(m+1)$. We have $\alpha(p) \sqsubset \alpha(p-1)$. Define $\delta := \min(\alpha'(p) - \alpha'(p-1), \alpha''(p-1) - \alpha''(p))$ and find $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. We claim this is the n we are looking for. Suppose $y \in I_2$ and $|x - y| < \frac{1}{n}$. Find $\beta \in \sigma_{\text{ir}}$ such that $\beta \equiv y$ and $\bar{\beta}p = \bar{\alpha}p$. This gives $f'(\beta)(m+1) = f'(\alpha)(m+1)$. Since $f'(\alpha) \in \sigma_{\text{reg}}$ we have $l(f'(\alpha)(m+1) \leq 2^{-m-1}$, which is easily shown with induction. Also $f'(\alpha)'(m+1) \leq f'(\alpha) \leq f'(\alpha)''(m+1)$ and $f'(\alpha)'(m+1) \leq f'(\beta) \leq f'(\alpha)''(m+1)$ and $f'(\alpha) \equiv f(x)$ and $f'(\beta) \equiv f(y)$ so $|f(x) - f(y)| \leq \frac{1}{m}$. To show that not every function $f : I_2 \to \mathbb{R}$ is uniformly continuous we consider the following counterexample: $f(x) = 1/x$. □

5.3 Examples 3 and 4

Brouwer defines the geometric type of J_1 [(4)] and the geometric type of J_1' [(5)] as two different types, where:

$$J_1 = I_2 \cup \mathbb{Q}$$

$$J_1' = I_1 \cup \mathbb{Q}'$$

Since, by lemma 5.2.3, they are actually of the same geometric type we define C_1 the be the geometric type of J_1. Furthermore, we define C_2 to be the geometric type of J_2 [(6)], C_3 to be the geometric type of J_3 [(7)] and C_4 to be of the geometric type of J_4 [(8)], where:

$$J_2 = I_2 \cup ([0, 1] \setminus I_2)$$

[(4)] This is I_1 from example 4 of Brouwers article.
[(5)] This is H_1 from example 3 of Brouwers article.
[(6)] This is H_2 from example 3 of Brouwers article.
[(7)] This is I_2 from example 4 of Brouwers article.
[(8)] This is I_3 from example 4 of Brouwers article.

$$J_3 = \mathbb{Q} \cup ([0,1] \setminus \mathbb{Q})$$
$$= \mathbb{Q} \cup \{x \mid x \in [0,1] \mid \neg(x \in \mathbb{Q})\}$$

$$J_4 = ([0,1] \setminus \mathbb{Q}) \cup ([0,1] \setminus ([0,1] \setminus \mathbb{Q}))$$
$$= \{x \mid x \in [0,1] \mid \neg(x \in \mathbb{Q})\} \cup \{x \mid x \in [0,1] \mid \neg\neg(x \in \mathbb{Q})\}$$

In a sense, these examples all come from $\mathbb{Q}$. Brouwer also studies similar examples constructed with $\mathbb{Q}'$, but by lemma 5.2.3 this is not necessary.

Lemma 5.3.1. *$J_1 \subseteq J_2$, but we can not prove $J_2 \subseteq J_1$.*

Proof. $J_1 \subseteq J_2$ is obvious. To show that we can not prove $J_2 \subseteq J_1$ we will use the real number r. We can not prove that this number is rational nor that it is irrational but, by lemma 1.3.2, we know $\neg\neg(r$ is rational). Thus we can not prove that $\frac{1}{2}+r$ is rational nor that it is irrational, but $\neg\neg(\frac{1}{2}+r$ is rational). This means we can not prove $\frac{1}{2}+r \in I_2$ and we can not prove $\frac{1}{2}+r \in \mathbb{Q}$. Also, for every $x \in [0,1]$, if x is rational then $\neg(x \in I_2)$. So if $\neg\neg(x$ is rational) then $\neg\neg\neg(x \in I_2)$ so $\neg(x \in I_2)$ thus $x \in ([0,1] \setminus I_2)$. So $\frac{1}{2}+r \in ([0,1] \setminus I_2)$ thus $\frac{1}{2}+r \in J_2$. □

Recall, for every $c = (c_n)_{n=0}^{\infty}$, $I_c = \{x \mid x \in [0,1] \mid \forall n \in \mathbb{N}[x \# c_n]\}$.

Lemma 5.3.2. *Every representative of C_1 is of the form $I_c \cup \{c_n | n \in \mathbb{N}\}$ for some countable sequence $c = (c_n)_{n=0}^{\infty}$ with:*

(i) $\forall n \in \mathbb{N}[c_n \in [0,1]]$

(ii) $\forall n \neq m \in \mathbb{N}[c_n \# c_m]$

(iii) $(c_n)_{n=0}^{\infty}$ *is dense in* $[0,1]$

(iv) $\exists n, m \in \mathbb{N}[c_n = 0 \wedge c_m = 1]$

Proof. Suppose C' is a representative of C_1, then $I_2 \cup \mathbb{Q} \sim C'$. This means there exists a uniformly continuous bijection $f : [0,1] \to [0,1]$ such that $f(I_2 \cup \mathbb{Q}) = C'$. Now find an enumeration $q_0, q_1, q_2, \ldots$ of $\mathbb{Q}$ and consider $c_0 := f(q_0), c_1 := f(q_1), c_2 := f(q_2), \ldots$. This sequence is a sequence such that conditions $(i) - (iv)$ and are satisfied and such that $C' = I_c \cup \{c_n | n \in \mathbb{N}\}$. In the proof of lemma 5.2.4 we can see that conditions $(i) - (iv)$ are satisfied. So we will now prove $f(I_2 \cup \mathbb{Q}) = I_c \cup \{c_n | n \in \mathbb{N}\}$, which proves $C' = I_c \cup \{c_n | n \in \mathbb{N}\}$. But $f(I_2 \cup \mathbb{Q}) = f(I_2) \cup f(\mathbb{Q})$. By the proof of lemma 5.2.4 we know $f(I_2) = I_c$ and obviously $f(\mathbb{Q}) = \{c_n | n \in \mathbb{N}\}$. □

Lemma 5.3.3. *Every representative of C_2 is of the form $I_c \cup ([0,1] \setminus I_c)$ for some countable sequence $c = (c_n)_{n=0}^{\infty}$ with:*

(i) $\forall n \in \mathbb{N}[c_n \in [0,1]]$

(ii) $\forall n \neq m \in \mathbb{N}[c_n \# c_m]$

(iii) $(c_n)_{n=0}^{\infty}$ *is dense in* $[0,1]$

(iv) $\exists n, m \in \mathbb{N}[c_n = 0 \wedge c_m = 1]$

Proof. Suppose C' is a representative of C_2, then $I_2 \cup ([0,1] \setminus I_2) \sim C'$. This means there exists a uniformly continuous bijection $f : [0,1] \to [0,1]$ such that $f(I_2 \cup ([0,1] \setminus I_2)) = C'$. Now find an enumeration $q_0, q_1, q_2, \dots$ of $\mathbb{Q}$ and consider $c_0 := f(q_0), c_1 := f(q_1), c_2 := f(q_2), \dots$. This sequence is a sequence such that conditions $(i) - (iv)$ are satisfied and such that $C' = I_c \cup ([0,1] \setminus I_c)$. In the proof of lemma 5.2.4 we can see that conditions $(i) - (iv)$are satisfied. So we will now prove $f(I_2 \cup ([0,1] \setminus I_2)) = I_c \cup ([0,1] \setminus I_c)$, which proves $C' = I_c \cup ([0,1] \setminus I_c)$. But $f(I_2 \cup ([0,1] \setminus I_c)) = f(I_2) \cup f([0,1] \setminus I_c)$. By the proof of lemma 5.2.4 we know $f(I_2) = I_c$ so obviously also $f([0,1] \setminus I_2) = [0,1] \setminus I_c$. □

Lemma 5.3.4. *Every representative of either C_1 or C_2 has measure 1.*

Proof. Suppose C' is a representative of C_1 or C_2. By lemma 5.3.2 and 5.3.3 we know C' is of the form $I_c \cup \{c_n | n \in \mathbb{N}\}$ or of the form $I_c \cup ([0,1] \setminus I_c)$ for some $c = (c_n)_{n=0}^{\infty}$. Thus it is enough to show I_c has measure 1. But we already showed this in the proof of lemma 5.2.5. □

Lemma 5.3.5. *Every representative of C_3 is of the form $\{c_n | n \in \mathbb{N}\} \cup ([0,1] \setminus (\{c_n | n \in \mathbb{N}\})$ for some countable sequence $c = (c_n)_{n=0}^{\infty}$ with:*

(i) $\forall n \in \mathbb{N}[c_n \in [0,1]]$

(ii) $\forall n \neq m \in \mathbb{N}[c_n \# c_m]$

(iii) $(c_n)_{n=0}^{\infty}$ *is dense in* $[0,1]$

(iv) $\exists n, m \in \mathbb{N}[c_n = 0 \wedge c_m = 1]$

Proof. Suppose C' is a representative of C_3, then $\mathbb{Q} \cup ([0,1] \setminus \mathbb{Q}) \sim C'$. This means there exists a uniformly continuous bijection $f : [0,1] \to [0,1]$ such that $f(\mathbb{Q} \cup ([0,1] \setminus \mathbb{Q})) = C'$. Now find an enumeration $q_0, q_1, q_2, \dots$ of $\mathbb{Q}$ and consider $c_0 := f(q_0), c_1 := f(q_1), c_2 := f(q_2), \dots$. This sequence is a sequence such that conditions $(i) - (iv)$ are satisfied and such that $C' = \{c_n | n \in \mathbb{N}\} \cup ([0,1] \setminus \{c_n | n \in \mathbb{N}\})$. In the proof of lemma 5.2.4 we can see that conditions $(i) - (iv)$ are satisfied. So we will now prove $f(\mathbb{Q} \cup ([0,1] \setminus \mathbb{Q})) = \{c_n | n \in \mathbb{N}\} \cup ([0,1] \setminus \{c_n | n \in \mathbb{N}\})$, which proves $C' = \{c_n | n \in \mathbb{N}\} \cup ([0,1] \setminus \{c_n | n \in \mathbb{N}\})$. But $f(\mathbb{Q} \cup ([0,1] \setminus \mathbb{Q})) = f(\mathbb{Q}) \cup f([0,1] \setminus \mathbb{Q})$. Obviously we have $f(\mathbb{Q}) = \{c_n | n \in \mathbb{N}\}$. This immediately implies $f([0,1] \setminus \mathbb{Q}) = [0,1] \setminus \{c_n | n \in \mathbb{N}\}$. □

Lemma 5.3.6. *Every representative of C_4 is of the form $([0,1] \setminus \{c_n | n \in \mathbb{N}\}) \cup ([0,1] \setminus ([0,1] \setminus (\{c_n | n \in \mathbb{N}\}))$ for some countable sequence $c = (c_n)_{n=0}^{\infty}$ with:*

(i) $\forall n \in \mathbb{N}[c_n \in [0,1]]$

(ii) $\forall n \neq m \in \mathbb{N}[c_n \# c_m]$

(iii) $(c_n)_{n=0}^{\infty}$ *is dense in* $[0,1]$

(iv) $\exists n, m \in \mathbb{N}[c_n = 0 \wedge c_m = 1]$

Proof. Suppose C' is a representative of C_4, then $([0,1] \setminus \mathbb{Q}) \cup ([0,1] \setminus ([0,1] \setminus \mathbb{Q})) \sim C'$. This means there exists a uniformly continuous bijection $f : [0,1] \to [0,1]$ such that $f(([0,1] \setminus \mathbb{Q}) \cup ([0,1] \setminus ([0,1] \setminus \mathbb{Q}))) = C'$. Now find an enumeration $q_0, q_1, q_2, \dots$ of $\mathbb{Q}$ and consider $c_0 := f(q_0), c_1 := f(q_1), c_2 := f(q_2), \dots$. This sequence is a sequence such that conditions $(i) - (iv)$ are satisfied and such that $C' = ([0,1] \setminus \{c_n | n \in \mathbb{N}\}) \cup ([0,1] \setminus ([0,1] \setminus \{c_n | n \in \mathbb{N}\}))$. In the proof of lemma 5.2.4 we can see that conditions $(i) - (iv)$ are satisfied. So we will now prove $f([0,1] \setminus \mathbb{Q} \cup [0,1] \setminus ([0,1] \setminus \mathbb{Q})) = ([0,1] \setminus \{c_n | n \in \mathbb{N}\}) \cup ([0,1] \setminus ([0,1] \setminus \{c_n | n \in \mathbb{N}\}))$, which proves $C' = ([0,1] \setminus \{c_n | n \in \mathbb{N}\}) \cup ([0,1] \setminus ([0,1] \setminus \{c_n | n \in \mathbb{N}\}))$. But $f(([0,1] \setminus \mathbb{Q}) \cup ([0,1] \setminus ([0,1] \setminus \mathbb{Q}))) = f([0,1] \setminus \mathbb{Q}) \cup f([0,1] \setminus ([0,1] \setminus \mathbb{Q}))$. Obviously we have $f(\mathbb{Q}) = \{c_n | n \in \mathbb{N}\}$. This immediately implies $f([0,1] \setminus \mathbb{Q}) = [0,1] \setminus \{c_n | n \in \mathbb{N}\}$ and $f([0,1] \setminus ([0,1] \setminus \mathbb{Q})) = [0,1] \setminus ([0,1] \setminus \{c_n | n \in \mathbb{N}\})$. □

Lemma 5.3.7. *Every representative of C_3 and C_4 had measure 1.*

Proof. Suppose C' is a representative of C_3 or C_4. By lemma 5.3.5 and 5.3.6 we know C' is of the form $\{c_n|n \in \mathbb{N}\} \cup ([0,1] \setminus \{c_n|n \in \mathbb{N}\})$ or of the form $([0,1] \setminus \{c_n|n \in \mathbb{N}\}) \cup ([0,1] \setminus ([0,1] \setminus \{c_n|n \in \mathbb{N}\}))$. Thus it is enough to show that $([0,1] \setminus \{c_n|n \in \mathbb{N}\})$ has measure 1. Following almost the same proof as the proof of lemma 5.2.5 we can prove this. □

Lemma 5.3.8. *(i) Every representative of C_1, C_2, C_3 and C_4 is not apart from $[0,1]$.*

(ii) Every representative of C_1, C_2, C_3 and C_4 does not coincide with $[0,1]$.

(iii) Every representative of C_2, C_3 and C_4 does not deviate from $[0,1]$.

Proof. By lemma 4.2 it is enough to show $(i), (ii)$ and (iii) for J_1, J_2, J_3 and J_4.

(i) First we will prove $\neg[J_1 \# [0,1]]$. Suppose there exists a $j \in J_1$ such that $j \# [0,1]$ or there exists an $x \in [0,1]$ such that $x \# J_1$. Then there exists an $x \in [0,1]$ such that $x \# J_1$, since $J_1 \subseteq [0,1]$. So there exists an $x \in [0,1]$ such that for every $j \in J_1$ $[x \# j]$, which means for every $j \in \mathbb{Q}'$ $[x \# j]$, so $x \in I_2$. Since also for every $j \in I_2[x \# j]$ this will give $[x \# x]$, which is a contradiction.
Secondly we will prove $\neg[J_2 \# [0,1]]$. Suppose there exists a$j \in J_2$ such that $j \# [0,1]$ or there exists an $x \in [0,1]$ such that $x \# J_2$. Then there exists an $x \in [0,1]$ such that $x \# J_2$, since $J_2 \subseteq [0,1]$. So there exists an $x \in [0,1]$ such that for every $j \in J_2$ $[x \# j]$, which means for every $j \in I_1[x \# j]$, so $\neg(x \in I_1)$ and thus $x \in ([0,1] \setminus I_2)$. Since also for every $j \in ([0,1] \setminus I_2)[j \# x]$ this will give $[x \# x]$, which is a contradiction.
Now we will prove $\neg[J_3 \# [0,1]]$. Suppose there exists a $j \in J_3$ such that $j \# [0,1]$ or there exists an $x \in [0,1]$ such that $x \# J_3$. Then there exists an $x \in [0,1]$ such that $x \# J_3$, since $J_3 \subseteq [0,1]$. So there exists an $x \in [0,1]$ such that for every $j \in J_3, x \# j$, which means for every $j \in \mathbb{Q}, x \# j$, so $x \notin \mathbb{Q}$ so $x \in ([0,1] \setminus \mathbb{Q})$. Since also for every $j \in ([0,1] \setminus \mathbb{Q}), x \# j$ this will give $x \# x$, which is a contradiction.
Lastly we will prove $\neg[J_4 \# [0,1]]$. Suppose there exists a $j \in J_4$ such that $j \# [0,1]$ or there exists an $x \in [0,1]$ such that $x \# J_4$. Then there exists an $x \in [0,1]$ such that $x \# J_4$, since $J_4 \subseteq [0,1]$. So there exists an $x \in [0,1]$ such that for every $j \in J_4, x \# j$, which means for every $j \in ([0,1] \setminus \mathbb{Q}), x \# j$, so $x \notin ([0,1] \setminus \mathbb{Q})$ so $x \in ([0,1] \setminus ([0,1] \setminus \mathbb{Q}))$. Since also for every $j \in ([0,1] \setminus ([0,1] \setminus \mathbb{Q})), x \# j$ this will give $x \# x$, which is a contradiction.

(ii) This follows directly from lemma 2.2.8.

(iii) Firstly we prove $\neg[J_2 \not\equiv [0,1]]$. So we have to prove $\neg\big[\exists j \in J_2\ \neg[j \in_0 [0,1]]\big]$ and $\neg\big[\exists x \in [0,1]\ \neg[x \in_0 J_2]\big]$. First we will prove $\neg\big[\exists j \in J_2\ \neg[j \in_0 [0,1]]\big]$. This is trivial since $J_2 \subseteq [0,1]$, so for every $j \in J_2$, $j \in_0 [0,1]$. Now we will prove $\neg\big[\exists x \in [0,1]\ \neg[x \in_0 J_2]\big]$. Suppose there exists $x \in [0,1]$ such that $\neg(x \in_0 J_2$, then $\neg(x \in_0 I_2)$ so $\neg(x \in I_2)$. This means $x \in ([0,1] \setminus I_2)$, but also $\neg(x \in_0 ([0,1] \setminus I_2)$, which is a contradiction.
Secondly will prove $\neg[J_3 \not\equiv [0,1]]$. So we have to proof $\neg\big[\exists j \in J_3\ \neg[j \in_0 [0,1]]\big]$ and $\neg\big[\exists x \in [0,1]\ \neg[x \in_0 J_3]\big]$. First we will prove $\neg\big[\exists j \in J_3\ \neg[j \in_0 [0,1]]\big]$. This is trivial since $J_3 \subseteq [0,1]$, so $\forall j \in J_3\ [j \in_0 [0,1]]$. Now we will prove $\neg\big[\exists x \in [0,1]\ \neg[x \in_0 J_3]\big]$. Suppose $\big[\exists x \in [0,1]\ \neg[x \in_0 J_3]\big]$, then find this x. So $\neg(x \in_0 J_3])$, thus $\neg(x \in J_3)$ so $\neg(x \in \mathbb{Q})$ which means $x \in [0,1] \setminus \mathbb{Q}$. This means $x \in J_3$ so $x \in_0 J_3$, which is a contradiction.
Lastly we will prove $\neg[J_4 \not\equiv [0,1]]$. So we have to proof $\neg\big[\exists j \in J_4\ \neg[j \in_0 [0,1]]\big]$ and $\neg\big[\exists x \in [0,1]\ \neg[x \in_0 J_4]\big]$. First we will prove $\neg\big[\exists j \in J_4\ \neg[j \in_0 [0,1]]\big]$. This is trivial since $J_4 \subseteq [0,1]$, so $\forall j \in J_4\ [j \in_0 [0,1]]$. Now we will prove $\neg\big[\exists x \in [0,1]\ \neg[x \in_0 J_4]\big]$.

Suppose $\left[\exists x \in [0,1]\ \neg[x \in_0 J_4]\right]$, then find this x. So $\neg(x \in_0 J_4])$, thus $\neg(x \in J_4)$ so $\neg(x \in [0,1] \setminus \mathbb{Q})$ which means $x \in [0,1] \setminus ([0,1] \setminus \mathbb{Q})$. This means $x \in J_4$ so $x \in_0 J_4$, which is a contradiction.

□

Lemma 5.3.9. *For each representative C of C_1 or C_2 we can find a countable sequence $c_1, c_2, c_3, \dots \in [0,1]$ such that we can not prove $c_1, c_2, c_3, \dots \in C$ and such that $c_i \neq c_j$ for each $i \neq j$.*

Proof. By lemma 4.11 it is enough to show this for J_1 and J_2. First we will show this for J_1. Define $c_1, c_2, c_3, \dots$ with $c_i = 2^{-i} + r$ with r as above. Obviously $c_i \neq c_j$ for all $i \neq j$. Furthermore, pick $i \in \mathbb{N}$. Since we can not prove r is rational we can not prove $2^{-i} + r$ is rational, so we can not prove $2^{-i} + r \in \mathbb{Q}$. Also, we can not prove r is irrational so we can not prove $2^{-i}+r$ is irrational, so we can not prove $2^{-i}+r \in I_2$. Thus we can not prove $2^{-i}+r \in J_1$. We can do something similar for J_2, by taking a sequence $b_1, b_2, b_3, \dots$ of elements from I_2 such that $b_i \neq b_j$ for every $i \neq j$ and then define $c_i = b_i + r$ for every $i \in \mathbb{N}$. □

Lemma 5.3.10. *For each representative C of C_1, C_3 or C_4 we can find an in $[0,1]$ dense set X such that for all $x \in X$ we can not prove $x \in C$ and such that $\forall x_1, x_2 \in X$ we have $x_1 = x_2$ or $x_1 \# x_2$.*

Proof. By lemma 4.12 it is enough to show this for J_1, J_3 and J_4. We define $X_{1,3} = \{\ x \mid x \in [0,1] \mid x = a + r\ , a \in \mathbb{Q}\}$ and $X_4 = \{\ x \mid x \in [0,1] \mid x = a + \pi r\ , a \in \mathbb{Q}\}$.

(i) $X_{1,2}$ proves the claim for J_1 and J_3

(ii) X_4 proves the claim for J_4

We will now prove (i) and (ii).

(i) Since $\mathbb{Q}$ is dense in $\mathbb{R}$ we have $X_{1,3}$ is dense in $[0,1]$. Pick $x \neq y \in X_{1,3}$, then $x = a + r$ and $y = b + r$ with $a \neq b \in \mathbb{Q}$. So obviously $x \# y$. Now suppose $x \in X_{1,2}$, then $x = a + r$ with $a \in \mathbb{Q}$. Since we can not prove that r is rational or that r is irrational we can not prove that $a + r$ is rational or that $a + r$ is irrational. Thus we can not prove $a + r \in \mathbb{Q}$ and we can not prove $a + r \in ([0,1] \setminus \mathbb{Q})$. Also, suppose we prove $a + r \in I_2$ then we prove $a + r \# q$ for every $q \in \mathbb{Q}$. This means $a + r \notin \mathbb{Q}$, which we can not prove.

(ii) Since $\mathbb{Q}$ is dense in $\mathbb{R}$ we have X_4 is dense in $[0,1]$. Pick $x \neq y \in X_4$, then $x = a + \pi r$ and $y = b + \pi r$ with $a \neq b \in \mathbb{Q}$. So obviously $x \# y$. Now suppose $x \in X_4$, then $x = a + \pi r$ with $a \in \mathbb{Q}$. Suppose we prove $a + \pi r \in ([0,1] \setminus \mathbb{Q})$ then we prove $a + pir$ is not rational. This means we prove πr is not rational so we prove $r \neq 0$, but we can not prove that. Suppose we prove $a + \pi r \in \big([0,1] \setminus ([0,1] \setminus \mathbb{Q})\big)$, then we prove $\neg\neg(a + \pi r$ is rational). This means we prove $\neg\neg(\pi r$ is rational) so we prove $r = 0$, but we can not prove that.

□

Lemma 5.3.11. *For every representative C of C_1, C_2, C_3 or C_4 there exists a function $f : C \to \mathbb{R}$ such that f is discontinuous.*

Proof. By corollary 4.4 it is enough to show this for J_1, J_2, J_3 and J_4.

(i) Define a function $f : J_1 \to \mathbb{R}$ by $f(c) = 0$ if $c \in I_2$ and $f(c) = 1$ if $c \in \mathbb{Q}$. To prove f is discontinuous we consider any $x \in I_2$ and $n = 2$. Since $\forall q \in \mathbb{Q} \exists m \in \mathbb{N} |x - q| \geq \frac{1}{m}$ we can find $a_1, a_2, a_3, \dots$ such that $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$. Pick any $m \in \mathbb{N}$ and find $k \in \mathbb{N}$ such that $\sum_{n=k+1}^{\infty} \frac{1}{2^n} \leq \frac{1}{2^m}$. Define $q = \sum_{n=1}^{k} \frac{a_n}{2^n} \in \mathbb{Q}$. Then $|q - x| \leq \sum_{n=k+1}^{\infty} \frac{a_n}{2^n} \leq \sum_{n=k+1}^{\infty} \frac{1}{2^n} \leq \frac{1}{2^m}$ but $|f(q) - f(x)| = 1 > \frac{1}{4}$.

(ii) Define a function $f : J_2 \to \mathbb{R}$ by $f(c) = 0$ if $c \in I_2$ and $f(c) = 1$ if $c \in ([0,1] \setminus I_2)$. Since $\mathbb{Q} \subseteq ([0,1] \setminus I_2)$ the proof of (i) also applies to this function.

(iii) Define a function $f : J_3 \to \mathbb{R}$ by $f(c) = 0$ if $c \in \mathbb{Q}$ and $f(c) = 1$ if $c \in ([0,1] \setminus \mathbb{Q})$. Since $I_2 \subseteq ([0,1] \setminus \mathbb{Q})$ the proof of (i) also applies to this function.

(iv) Define a function $f : J_4 \to \mathbb{R}$ by $f(c) = 0$ if $c \in ([0,1] \setminus \mathbb{Q})$ and $f(c) = 1$ if $c \in ([0,1] \setminus ([0,1] \setminus \mathbb{Q}))$. Since $I_2 \subseteq ([0,1] \setminus \mathbb{Q})$ and $\mathbb{Q} \subseteq ([0,1] \setminus ([0,1] \setminus \mathbb{Q}))$ the proof of (i) also applies to this function.

□

5.4 Example 5

We define E to be the geometric type of L [9], where $L = L' \cup L''$ and:

$$L' = \{\ x \bigm| x \in [0,1] \bigm| x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mid \forall n \in \mathbb{N}[a_n \in \{0,2\}]\ \}$$

$$\begin{aligned} L'' = \{\ x \bigm| x \in [0,1] \bigm| x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mid \exists m \in \mathbb{N}\ \big[&\forall n < m[a_n \in \{0,2\}]\ \wedge\ a_m = 1\ \wedge \\ &\forall n > m[a_n \in \{0,1,2\}] \wedge \\ &\exists\ p, q > m[a_p \neq 0\ \wedge\ a_q \neq 2]\big]\} \end{aligned}$$

So L' is the Cantor discontinuum and L'' are all the $x \in [0,1]$ which we can write as $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ for some sequence $a_1, a_2, a_3, \dots \in \{0,1,2\}$ and which are in the union of open intervals in the complement of the Cantor discontinuum. Brouwer seems to think that L'' is the union of the all the open intervals in the complement of the Cantor discontinuum, but this is not true. We can not prove that we can write every $x \in [0,1]$ as $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ for some sequence $a_1, a_2, a_3, \dots \in \{0,1,2\}$.

Lemma 5.4.1. $L' = [0,1] \setminus L''$

Note that we can not prove $L'' = [0,1] \setminus L'$.

Proof. Suppose $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \in L'$. This means $\forall n \in \mathbb{N}\ [a_n \in \{0,2\}]$. So $\neg\big[\exists m \in \mathbb{N}\ [a_m = 1]\big]$, so $x \in [0,1] \setminus L''$.
Suppose $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \in [0,1] \setminus L''$. This means there does not exists an $m \in \mathbb{N}$ such that

[9] This is J from example 5 of Brouwers article.

$a_m = 1$ or there exists $m \in \mathbb{N}$ such that $a_m = 1$ but there does not exists $p, g > m$ such that $a_p \neq 0$ and $a_q \neq 2$. Suppose there does not exists an $m \in \mathbb{N}$ such that $a_m = 1$, then clearly for all $n \in \mathbb{N}$, $a_n \in \{0, 2\}$. Suppose there exists $m \in \mathbb{N}$ such that $a_m = 1$ but there does not exists $p, q > m$ such that $a_p \neq 0$ and $a_q \neq 2$. This means for all $p > m$, $a_p \neq 1$. Find $m \in \mathbb{N}$ such that $a_m = 1$ and consider $m + 1$. Suppose $a_{m+1} = 0$ (the case where $a_{m+1} = 2$ is similar), then there exists $q > m$ such that $a_q \neq 2$ and thus there does not exists a $p > m$ such that $a_p \neq 0$. This means for all $p > m, a_p = 0$. Then $x = \sum_{n=1}^{\infty} \frac{b_n}{3^n}$ with $b_n = a_n$ for all $n < m$, $b_m = 0$ and $b_n = 2$ for all $n > m$. □

Lemma 5.4.2. *Every representative of E is of the form $X' \cup X''$ for some $X', X'' \subseteq [0, 1]$ such that $X' = [0, 1] \setminus X''$.*

Proof. Suppose E' is a representative of E then $L' \cup L'' \sim E'$. So there exists a uniformly continuous bijection $f : [0, 1] \to [0, 1]$ such that $f(L' \cup L'') = E'$. But $f(L' \cup L'') = f(L') \cup f(L'')$. Define $X' = f(L')$ and $X'' = f(L'')$. Now suppose $x \in X'$, then $f^{-1}(x) \in L'$ so $f^{-1}(x) \notin L''$ thus $x \notin X''$. Suppose $x \notin X''$ then $f^{-1}(x) \notin L''$ so $f^{-1}(x) \in L'$ thus $x \in X'$. □

We will prove that L' is measurable and the measure of L' is 0. This means L'' is measurable and $\mu(L'') = 1$.

Lemma 5.4.3. *L' is measurable and $\mu(L') = 0$.*

Proof. We will prove that $\chi_{L'}$ is measurable. We will define an infinite sequence $X_0, X_1, X_2, \ldots$ of measurable regions and an infinite sequence $v_0, v_1, v_2, \ldots$ of elementary sets of rectangles such that they satisfy (i) and (ii) of definition 3.3.1.
First, for every $m \geq 1$ we will define a measurable region Y_m. We select a subsequence of the sequence $Y_1, Y_2, Y_3, \ldots$ to define our sequence $X_0, X_1, X_2, \ldots$.
Pick $m \geq 1$ and define $L^m := \{x \in [0, 1] \mid x = \sum_{n=1}^{m} \frac{a_n}{3^n} \mid \forall n \leq m[a_n \in \{0, 2\}]\}$. The number of elements in L^m is 2^m. We enumerate L^m with $q_0^m, q_1^m, \ldots, q_{2^m-1}^m$. Now define $Y_m := \mathcal{R}(\alpha_m(0), \alpha_m(1), \alpha_m(2), \ldots)$ where $\alpha_m(i) = (q_i^m - \frac{\frac{1}{2}}{3^{m-1}}, q_i^m + \frac{\frac{1}{2}}{3^{m-1}})$ for all $i < 2^m$ and $\alpha_m(i) = (q_{2^m-1}^m - \frac{\frac{1}{2}}{3^{m-1}}, q_{2^m-1}^m + \frac{\frac{1}{2}}{3^{m-1}})$ for all $i \geq 2^m$.
We will prove, for every $x \in [0, 1]$, if $x \notin Y_m$ then $x \in [0, 1] \setminus L' \subseteq \text{dom}(\chi_{L'})$. Suppose $x \in [0, 1]$ and suppose $x \notin Y_m$. Also suppose $x \in L'$, then $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ with $a_n \in \{0, 2\}$ for all $n \geq 1$. Consider $q = \sum_{n=1}^{m} \frac{a_n}{3^n}$. Then $q \in L^m$ and $x \geq q$ which means $x > q - \frac{\frac{1}{2}}{3^{m-1}}$. Also $x \leq q + \frac{1}{3^m} < q + \frac{\frac{1}{2}}{3^{m-1}}$. So $x \; \varepsilon_1 \; (q - \frac{\frac{1}{2}}{3^{m-1}}, q + \frac{\frac{1}{2}}{3^{m-1}})$ which means $x \in Y_m$. This is a contradiction, so $x \in [0, 1] \setminus L'$. Also, $\mu(Y_m) \leq \frac{2^m}{3^{m-1}} = 3(\frac{2}{3})^m$.
Now, for every $n \in \mathbb{N}$, find $m \in \mathbb{N}$ such that $3(\frac{2}{3})^m \leq \frac{1}{2^n}$ and define $X_n = Y_m$. So, for every $n \in \mathbb{N}$, $\mu(Y_n) \leq \frac{1}{2^n}$ and if $x \notin Y_n$ then $x \in \text{dom}(\chi_{L'})$.
Now pick $n \in \mathbb{N}$. We define $v_n = (v_n)_0, (v_n)_1, \ldots, (v_n)_n$ with $((v_n)_i)_0 = (\frac{i}{n+1}, \frac{i+1}{n+1})$ and $((v_n)_i)_1 = (0, \frac{1}{2^n})$ for every $i \leq n$. Clearly $\text{Ar}^*(v_n) \leq \frac{1}{2^n}$. Now suppose $x \notin X_n$. By the above we know $x \in [0, 1] \setminus L'$ so $\chi_{L'}(x) = 0$. Furthermore $0 \leq 0 \leq \frac{1}{2^n}$ so $\chi_{L'}(x) \; \varepsilon_0 \; ((v_n)_i)_1$. So L' is measurable and $\mu(L') = \lim_{n \to \infty} \sum_{i=0}^{\text{length}(v_n)-1} 0 = 0$. □

The next lemma will prove that every measurable representative of E has measure 1. After that we will give a representative L_2 of E for which we can not prove it is measurable. This representative was not given by Brouwer, but we add it to show there exist representatives of E for which we can not show they are measurable.

Lemma 5.4.4. *Every measurable representative of E has measure 1.*

Proof. Suppose E' is a measurable representative of E. By lemma 5.4.2 E' is of the form $X' \cup X''$ for some $X', X'' \subseteq [0,1]$ such that $X' = [0,1] \setminus X''$. Since $X' \cup X''$ is measurable, by lemma 3.3.10, $(X' \cup X'') \cup ([0,1] \setminus (X' \cup X''))$ is almost full. But $(X' \cup X'') \cup ([0,1] \setminus (X' \cup X'')) = ([0,1] \setminus X'') \cup X'' \cup \{x \in [0,1] | x \notin [0,1] \setminus X'' \text{ and } x \notin X''\} = ([0,1] \setminus X'') \cup X''$. So $([0,1] \setminus X'') \cup X'' = \mathrm{dom}(\chi_{X''})$ is almost full. Also $\chi_{X''}$ is bounded, so by theorem 3.3.4, X'' is measurable. Suppose $\mu(X'') = k$. By theorem 3.3.15, $([0,1] \setminus X'')$ is measurable and $\mu([0,1] \setminus X'') = 1 - k$. Also, by part 2. of the proof of theorem 3.3.15, $\mu(X'' \cup ([0,1] \setminus X'')) = \mu(X'' \cup X') = 1$. □

Before we define a representative of E for which we can not prove it is measurable we need to define a representative L_1 of E which is measurable. This representative was also not given by Brouwer. We define $L_1 = L_1' \cup L_1''$, where:

$$L_1' = \{\ x \mid x \in [0,1] \mid x = \sum_{n=1}^{\infty} \frac{a_n}{6^n} \mid \forall n \in \mathbb{N}[a_n \in \{0, \frac{3^n+5}{2}\}]\ \}$$

$$\begin{aligned} L_1'' = \{\ x \mid x \in [0,1] \mid x = \sum_{n=1}^{k} \frac{a_n}{6^n} + \frac{2}{6^k} \sum_{m=1}^{\infty} \frac{b_m}{3^m} \mid & \forall n < k[a_k \in \{0, \frac{3^n+5}{2}\}] \wedge \\ & a_k = \frac{3^k+1}{2} \wedge \\ & \forall m \in \mathbb{N}[b_m \in \{0,1,2\}] \\ & \wedge\ \exists p, q \in \mathbb{N}[b_p \neq 0 \wedge b_q \neq 2]]\} \end{aligned}$$

So L_1' is an alternative discontinuum and L_1'' are all the $x \in [0,1]$ which we can write as $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ for some sequence $a_1, a_2, a_3, \dots \in \{0,1,2\}$ and which are in the union of the open intervals in the complement of L_1'. Intuitively, for L_1' we first 'delete' the middle interval of size $\frac{2}{6}$. Then, for every interval that is left (which are two intervals), we 'delete' the middle interval of size $\frac{2}{6^2}$. Then, for every interval that is left (which are four intervals), we 'delete' the middle interval of size $\frac{2}{6^3}$, etc. This means we first delete intervals of total size $\frac{1}{3}$, then intervals of total size $\frac{1}{9}$, then of total size $\frac{1}{27}$, etc. See figure 6.

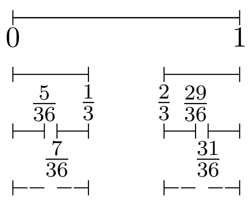

Figure 6: Alternative discontinuum.

We will now show L_1 is a representative of E.

Lemma 5.4.5. $L_1 \sim L$

Proof. We will define a total bijection $f : [0,1] \to [0,1]$. For this we define two sets $Q, Q' \subseteq [0,1]$ with:

$$Q := \{x \in [0,1] \mid \exists m \in \mathbb{N}\big[x = \sum_{n=1}^{m} \frac{a_n}{3^n} \ \wedge \ \forall n \leq m[a_n \in \{0,1,2\}]\ \big]\}$$

$$\begin{aligned} Q' :=&\{x \in [0,1] \mid \exists m \in \mathbb{N}\big[x = \sum_{n=1}^{m} \frac{a_n}{6^n} \ \wedge \ \forall n \leq m[a_n \in \{0, \frac{3^n+5}{2}\}]\ \big]\} \cup \\ &\{x \in [0,1] \mid \exists k, l \in \mathbb{N}\big[x = \sum_{n=1}^{k} \frac{a_n}{6^n} + \frac{2}{6^k} \sum_{m=1}^{l-k} \frac{b_m}{3^m} \ \wedge \\ &\qquad \forall n < k[a_n \ \in \{0, \frac{3^n+5}{2}\}] \ \wedge \ a_k = \frac{3^k+1}{2} \ \wedge \\ &\qquad \forall m \leq l-k[a_m \in \{0,1,2\}]\ \big]\} \end{aligned}$$

So Q are all the elements from L which we can write as a finite sum and Q' are all the elements from L_1 which we can write as a finite sum. We define a bijection $g : Q \to Q'$ and use this bijection to define f.

For every $q = \sum_{n=1}^{l} \frac{a_n}{3^n} \in Q$, define:

$$g(q) = \begin{cases} \sum_{n=1}^{l} \frac{b_n}{6^n} & \text{if } \forall n \leq l[a_n \neq 1] \\ \sum_{n=1}^{k} \frac{b_n}{6^n} + \frac{2}{6^k} \sum_{m=1}^{l-k} \frac{c_m}{3^m} & \text{if } \exists k \leq l[a_k = 1] \end{cases}$$

with, if $\forall n \leq l[a_n \neq 1]$ then, for all $n \leq l$, $b_n = 0$ if $a_n = 0$ and $b_n = \frac{3^n+5}{2}$ if $a_n = 2$ and if $\exists k \leq l[a_k = 1]$ then, for all $n < k$, $b_n = 0$ if $a_n = 0$ and $b_n = \frac{3^n+5}{2}$ if $a_n = 2$, $b_k = \frac{3^k+1}{2}$ and $\forall m \leq l-k$, $c_m = a_{m+k}$.

We will prove g is a bijection. Take $q = \sum_{n=1}^{l} \frac{a_n}{3^n}, q' = \sum_{n=1}^{l'} \frac{a'_n}{3^n} \in Q$ and decide $l \geq l'$ or $l' < l$. Suppose, without loss of generality, $l \geq l'$. Then define $q' = \sum_{n=1}^{l} \frac{a'_n}{3^n}$ with $a'_n = 0$ for all $l' < n \leq l$. Suppose $q \neq q'$. Find the smallest $k \leq l$ such that $a_k \neq a'_k$. Suppose $a_k = 0$ and $a'_k = 2$. Then $g(q) < {}^{k-1}_{n=1} \frac{a_n}{6^n} + \frac{\frac{3^k+1}{2}}{6^k}$ and $g(q') \geq {}^{k-1}_{n=1} \frac{a_n}{6^n} + \frac{\frac{3^k+5}{2}}{6^k}$. Suppose $a_k = 1$ and $a'_k = 2$. Then $g(q) < \sum_{n=1} k - 1 \frac{a_n}{6^n} + \frac{\frac{3^k+1}{2}}{6^k} + \frac{2}{6^k}$ and $g(q') \geq g(q) < \sum_{n=1} k - 1 \frac{a_n}{6^n} + \frac{\frac{3^k+5}{2}}{6^k}$. Suppose $a_k = 1$ and $a'_k = 0$. Then $g(q) \geq \sum_{n=1}^{k-1} \frac{a_n}{6^n} + \frac{\frac{3^k+1}{2}}{6^k}$ and $g(q') < \sum_{n=1}^{k-1} \frac{a_n}{6^n} + \frac{\frac{3^k+1}{2}}{6^k}$.

Now we can define f.

For every $x \in [0,1]$ there exists $y_x \in [0,1]$ such that $y_x \equiv x$ and $y_x = y_x(0), y_x(1), y_x(2), \ldots$ with, for every $i \in \mathbb{N}$, $y'_x(i), y''_x(i) \in Q$. Pick $x \in [0,1]$ and find y_x. Define $f(x) = (g(y'_x(0)), g(y''_x(0))), (g(y'_x(1)), g(y''_x(1$

Clearly, f is a uniformly continuous function. We will prove f is a bijection. Also, we will prove $f(L) = L_1$. This will prove $L_1 \sim L$.

First we will prove f is a bijection. Suppose $x, x' \in [0,1]$ and $x \# x'$. Find $y_x, y_{x'}$. Then $y_x \# y_{x'}$ so there exists $k \in \mathbb{N}$ such that $y''_x(k) < y'_{x'}(k)$ or $y''_{x'}(k) < y'_x(k)$. Suppose, without loss of generality, $y''_x(k) < y'_{x'}(k)$ then $g(y''_x(k)) < g(y'_{x'}(k))$ so $f(x)''(k) < f(x')'(k)$ which means $f(x) \# f(x')$. Now suppose $y \in [0,1]$, then there exist $z_y \in [0,1]$ such that $z_y \equiv y$ and $z_y = z_y(0), z_y(1), z_y(2), \ldots$ with, for every $i \in \mathbb{N}$, $z'_y(i), z''_y(i) \in Q'$. Now consider $x = (g^{-1}(z'_y(0)), g^{-1}(z''_y(0))), (g^{-1}(z'_y(1)), g^{-1}(z''_y(1))), (g^{-1}(z'_y(2)), g^{-1}((z''_y(2))), \ldots$. Then $f(x) = z_y$ so $f(x) \equiv y$, so $f(x) = y$.

Now we will prove $f(L) = L_1$. Suppose $x \in L$, then $x \in L'$ or $x \in L''$. Suppose $x \in L'$. First we note that $g(\{x \mid x \in Q \mid x \in L'\}) = \{x \mid x \in Q' \mid x \in L'_1\}$. Now, $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ such that $\forall n \in \mathbb{N}$, $a_n \in \{0,2\}$. Also $\lim_{m\to\infty} \sum_{n=1}^{m} \frac{a_n}{3^n} = x$. So, for all $k \in \mathbb{N}$ exists $m \in \mathbb{N}$ such that $|x - \sum_{n=1}^{m} \frac{a_n}{3^n}| \le \frac{1}{2^k}$. Pick $l \in \mathbb{N}$ and find $k \in \mathbb{N}$ such that for all $x, y \in [0,1]$ if $|x-y| \le \frac{1}{2^k}$ then $|f(x) - f(y)| \le \frac{1}{2^l}$. Find $m \in \mathbb{N}$ such that $|x - \sum_{n=1}^{m} \frac{a_n}{3^n}| \le \frac{1}{2^k}$. Then $|f(x) - f(\sum_{n=1}^{m} \frac{a_n}{3^n})| = |f(x) - g(\sum_{n=1}^{m} \frac{a_n}{3^n})| \le \frac{1}{2^k}$. So $\lim_{m\to\infty} g(\sum_{n=1}^{m} \frac{a_n}{3^n}) = f(x)$, so $f(x) \in L'_1$. A similar argument proves if $x \in L''$ then $f(x) \in L''_1$. □

Lemma 5.4.6. *L'_1 is measurable and $\mu(L'_1) = \frac{1}{2}$.*

Proof. We will define a measurable region X with $\mu(X) = \frac{1}{2}$ and prove $[0,1] \setminus X = L'_1$. Pick $M > 1$. Define $L^M := \{x \in [0,1] \mid x = \sum_{n=1}^{M-1} \frac{a_n}{6^n} \mid \forall n \le M-1 [a_n \in \{0, \frac{3^n+5}{2}\}]\}$. Also, define $L^1 = \{0\}$. Define $X_M = \bigcup_{x \in L^M} \left(x + \frac{\frac{3^M+1}{2}}{6^M}, x + \frac{\frac{3^M+5}{2}}{6^M}\right)$. Now define $X = \bigcup_{M \ge 1} X_M$. So X are actually all the open intervals in the complement of L'_1. This means $L''_1 \subseteq X$. Since we can not prove, for every $x \in [0,1]$ there exists a sequence $a_0, a_1, a_2, \dots \in \{0,1,2\}$ such that $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, we can not prove $X \subseteq L''_1$. But we can prove $L'_1 = [0,1] \setminus X$.

We will prove:

(i) $L'_1 \subseteq [0,1] \setminus X$

(ii) $[0,1] \setminus X \subseteq L'_1$

(iii) X is measurable and $\mu(X) = \frac{1}{2}$

With (i) and (ii) we prove $L'_1 = [0,1] \setminus X$. Combining this with (iii) we prove L'_1 is measurable and $\mu(L'_1) = \frac{1}{2}$. We will now prove (i), (ii) and (iii).

(i) Suppose $x \in L'_1$. Then $x = \sum_{n=1}^{\infty} \frac{a_n}{6^n}$ with $a_n \in \{0, \frac{3^n+5}{2}\}$ for all $n \ge 1$. Now suppose $x \in X$. Then find $M \ge 1$ and $y = \sum_{n=1}^{M-1} \frac{b_n}{6^n} \in L^M$ such that $x \; \varepsilon_1 \left(y + \frac{\frac{3^M+1}{2}}{6^M}, y + \frac{\frac{3^M+5}{2}}{6^M}\right)$. This means $y + \frac{\frac{3^M+5}{2}}{6^M} - x < \frac{2}{6^M}$. Now suppose $\exists m < M$ such that $a_m \neq b_m$. Find the smallest such m and suppose $a_m = 0$ and $b_m = \frac{3^m+5}{2}$. Then $x \le \sum_{n=1}^{m-1} \frac{b_n}{6^n} + \frac{\frac{3^m+1}{2}}{6^m}$ and

$y + \frac{\frac{3^M+5}{2}}{6^M} \geq y \geq \sum_{n=1}^{m-1} \frac{b_n}{6^n} + \frac{\frac{3^m+5}{2}}{6^m}$. This means $|x-y| \geq \frac{2}{6^M}$. So, for all $m < M$, $a_m = b_m$. Now suppose $a_M = 0$ then $x \leq y + \frac{\frac{3^M+1}{2}}{6^M}$, which is a contradiction. Suppose $a_M = 2$ then $x \geq y + \frac{\frac{3^M+5}{2}}{6^M}$, which is a contradiction. So $a_M = 1$, which is a contradiction. So $x \notin X$.

(ii) Suppose $x \notin X$. Now suppose $x \in L_1''$ then find $k, p, q \in \mathbb{N}$ such that $x = \sum_{n=1}^{k} \frac{a_n}{6^n} + \frac{2}{6^k} \sum_{m=1}^{\infty} \frac{b_m}{3^m}$ with for all $n < k$, $a_n \in \{0, \frac{3^n+5}{2}\}$, $a_k = \frac{3^m+1}{2}$, $b_p \neq 0$ and $b_q \neq 2$. Consider $y = \sum_{n=1}^{k-1} \frac{a_n}{6^n}$. Then, since $x \notin X$ we have $x \leq y + \frac{\frac{3^k+1}{2}}{6^k}$ or $x \geq y + \frac{\frac{3^k+5}{2}}{6^k}$. Suppose $x \leq y + \frac{\frac{3^k+1}{2}}{6^k}$, then $b_m = 0$ for all $m \in \mathbb{N}$, which is a contradiction. Suppose $x \geq y + \frac{\frac{3^k+5}{2}}{6^k}$ then $b_m = 2$ for all $m \in \mathbb{N}$, which is a contradiction. So $x \notin L''$ so $x \in L'$.

(iii) We will prove X is a measurable region and $\mu(X) = \frac{1}{2}$. $X = \bigcup_{M \geq 1} X_M$ where $X_M = \bigcup_{x \in L^M} \left(x + \frac{\frac{3^M+1}{2}}{6^M}, x + \frac{\frac{3^M+5}{2}}{6^M}\right)$. Define an enumeration $\alpha(1), \alpha(2), \alpha(3) \ldots$ of the intervals in X with $\alpha(i)$ is an interval of $X_{\lfloor 2\log(i)+1 \rfloor}$. $X = \mathcal{R}(\alpha)$.
We will prove $\lim_{n\to\infty} \mu(\bar{\alpha}n) = \frac{1}{2}$. For every $i \in \mathbb{N}$, $l(\alpha(i)) = \frac{2}{6^{\lfloor 2\log(i)+1 \rfloor}}$. Also, for every $i \neq j \in \mathbb{N}$, $\alpha(i)$ and $\alpha(j)$ are disjoint. So $\mu(\bar{\alpha}2^i - 1) = \sum_{n=1}^{2^i-1} \frac{2}{6^{\lfloor 2\log(i)+1 \rfloor}} = \sum_{n=1}^{i} \frac{2^n}{6^n}$. So $\lim_{i\to\infty} \mu(\bar{\alpha}2^i - 1) = \lim_{i\to\infty} \sum_{n=1}^{i} \frac{2^n}{6^n} = \frac{1}{2}$.

□

Now we can define the representative L_2 of E for which we can not prove it is measurable. We define $L_2 = L_2' \cup L_2''$, where:

$$L_2' = \Big\{x \;|x \in [0,1] \;|x = \sum_{n=1}^{k_1} \frac{a_n}{6^n} + \frac{\frac{3^{k_1}+1}{2}}{6^{k_1}} \sum_{m=1}^{\infty} \frac{b_m}{3^m}$$
$$\big|\forall n \leq k_1 \big[a_n \in \{0, \frac{3^n+5}{2}\}\big] \wedge \forall m \in \mathbb{N}\big[b_m \in \{0,2\}\big]\Big\}$$

$$
\begin{aligned}
L_2'' = \Big\{x \mid x \in [0,1] \mid \exists l \in \mathbb{N}\Big[l \leq k_1 \implies & \big[x = \sum_{n=1}^{l} \frac{a_n}{6^n} + \frac{2}{6^l} \sum_{m=1}^{\infty} \frac{b_m}{3^m} \wedge \\
& \forall n < l[a_n \in \{0, \frac{3^n+5}{2}\}] \wedge a_l = \frac{3^l+1}{2} \wedge \\
& \forall m \in \mathbb{N}[b_m \in \{0,1,2\}] \wedge \\
& \exists p, q \in \mathbb{N}[b_p \neq 0 \wedge b_q \neq 2]] \wedge \\
l > k_1 \implies & \big[x = \sum_{n=1}^{k_1} \frac{a_n}{6^n} + \frac{\frac{3^{k_1}+1}{2}}{6^{k_1}} \sum_{m=1}^{\infty} \frac{b_m}{3^m} \wedge \\
& \forall n \leq k_1[a_n \in \{0, \frac{3^n+5}{2}\}] \wedge \\
& \forall m \leq l[a_m \in \{0,2\}] \wedge a_l = 1 \wedge \\
& \forall m > l[b_m \in \{0,1,2\}] \wedge \\
& \exists p, q > l[b_p \neq 0 \wedge b_q \neq 2]]\Big]\Big\}
\end{aligned}
$$

So if $\neg\exists n \in \mathbb{N}[n = k_1]$ then $L_2' = L_1'$ and $L_2'' = L_1''$. If $\exists n \in \mathbb{N}[n = k_1]$ then L_2' are 'small Cantor discontinua' inside the intervals left at step k_1 and L_2'' are again all the real numbers in the union of the open intervals in the complement of L_2' such that $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ for a sequence $a_1, a_2, a_2, \dots \in \{0,1,2\}$.
First we will show L_2 is a representative of E.

Lemma 5.4.7. $L_2 \sim L_1$

Proof. This will be similar to the proof of lemma 5.4.5.
We will define a total bijection $f : [0,1] \to [0,1]$. For this we define two sets $Q, Q' \subseteq [0,1]$ with:

$$
\begin{aligned}
Q := & \{x \in [0,1] \mid \exists m \in \mathbb{N}\big[x = \sum_{n=1}^{m} \frac{a_n}{6^n} \wedge \forall n \leq m[a_n \in \{0, \frac{3^n+5}{2}\}]\big]\} \cup \\
& \{x \in [0,1] \mid \exists k, l \in \mathbb{N}\big[x = \sum_{n=1}^{k} \frac{a_n}{6^n} + \frac{2}{6^k} \sum_{m=1}^{l-k} \frac{b_m}{3^m} \wedge \\
& \qquad \forall n < k[a_n \in \{0, \frac{3^n+5}{2}\}] \wedge a_k = \frac{3^k+1}{2} \wedge \\
& \qquad \forall m \leq l-k[a_m \in \{0,1,2\}]\big]\}
\end{aligned}
$$

$$
\begin{aligned}
Q' :=\Big\{ x \;|x \in [0,1] \;|\exists l \in \mathbb{N}[x = \sum_{n=1}^{k_1} \frac{a_n}{6^n} + \frac{\frac{3^{k_1}+1}{2}}{6^{k_1}} \sum_{m=1}^{l-k_1} \frac{b_m}{3^m}] \\
\big|\forall n \le k_1 \big[a_n \in \{0, \frac{3^n+5}{2}\}\big] \wedge \forall m \le l - k_1 \big[b_m \in \{0,2\}\big]\Big\} \cup \\
\Big\{ x \;|x \in [0,1] \;|\exists l \in \mathbb{N} \exists l' \le l \Big[l' \le k_1 \implies \big[x = \sum_{n=1}^{l'} \frac{a_n}{6^n} + \frac{2}{6^{l'}} \sum_{m=1}^{l-l'} \frac{b_m}{3^m} \wedge \\
\forall n < l'[a_n \in \{0, \frac{3^n+5}{2}\}] \;\wedge\; a_{l'} = \frac{3^{l'}+1}{2} \wedge \\
\forall m \le l - l'[b_m \in \{0,1,2\}]\big] \wedge \\
l' > k_1 \implies \big[x = \sum_{n=1}^{k_1} \frac{a_n}{6^n} + \frac{\frac{3^{k_1}+1}{2}}{6^{k_1}} \sum_{m=1}^{l-k_1} \frac{b_m}{3^m} \wedge \\
\forall n \le k_1[a_n \in \{0, \frac{3^n+5}{2}\}] \wedge \\
\forall m < l' - k_1[a_m \in \{0,2\}] \wedge \; a_{l'-k_1} = 1 \wedge \\
\forall m > l' - k_1[b_m \in \{0,1,2\}]\Big\}
\end{aligned}
$$

So Q are all the elements from L_1 which we can write as a finite sum and Q' are all the elements from L_2 which we can write as a finite sum. We define a bijection $g : Q \to Q'$ and use this bijection to define f.

Suppose $q \in Q$ and suppose exists $l \in \mathbb{N}$ with $q = \sum_{n=1}^{l} \frac{a_n}{6^n}$ with $a_n \in \{0, \frac{3^n+5}{3}\}$ for all $n \le l$, then define $g(q) = \sum_{n=1}^{k_1} \frac{a_n}{6^n} + \frac{\frac{3^{k_1}+1}{2}}{6^{k_1}} \sum_{m=1}^{l-k_1} \frac{b_m}{3^m}$ with $b_m = 0$ if $a_{k_1+m} = 0$ and $b_m = 2$ if $a_{k_1+m} = \frac{3^{k_1+m}+5}{2}$.

Suppose $q \in Q$ and suppose exists $k, l \in \mathbb{N}$ with $q = \sum_{n=1}^{k} \frac{a_n}{6^n} + \frac{2}{6^k} \sum_{m=1}^{l-k} \frac{b_m}{3^m}$ such that for all $n < k$, $a_n \in \{0, \frac{3^n+5}{2}\}$, $a_k = \frac{3^k+1}{2}$ and for all $m \le l - k$, $a_m \in \{0,1,2\}$, then:

$$
g(q) = \begin{cases} q & \text{if } k \le k_1 \\ \displaystyle\sum_{n=1}^{k_1} \frac{a_n}{6^n} + \frac{\frac{3^{k_1}+1}{2}}{6^{k_1}} \sum_{m=1}^{l-k} \frac{b_m}{3^m} & \text{if } k > k_1 \end{cases}
$$

with, for all $m \le k$, $b_m = 0$ if $a_m = 0$ and $b_m = 2$ if $a_m = \frac{3^m+5}{2}$.
Similar, but a bit more complicated, to the proof of 5.4.5 we can prove g is a bijection. Now we can define f.
For every $x \in [0,1]$ there exists $y_x \in [0,1]$ such that $y_x \equiv x$ and $y_x = y_x(0), y_x(1), y_x(2), \ldots$ with, for every $i \in \mathbb{N}$, $y'_x(i), y''_x(i) \in Q$. Pick $x \in [0,1]$ and find y_x. Define $f(x) = (g(y'_x(0)), g(y''_x(0))), (g(y'_x(1)), g(y''_x(1)$
Clearly, f is a total function. This means, by the uniform continuity theorem, f is uniformly continuous. We will prove f is a bijection. Also, we will prove $f(L_1) = L_2$. This will prove $L_1 \sim L_2$.
First we will prove f is a bijection. Suppose $x, x' \in [0,1]$ and $x \# x'$. Find $y_x, y_{x'}$. Then $y_x \# y_{x'}$ so there exists $k \in \mathbb{N}$ such that $y''_x(k) < y'_{x'}(k)$ or $y''_{x'}(k) < y'_x(k)$. Suppose, without loss of generality, $y''_x(k) < y'_{x'}(k)$ then $g(y''_x(k)) < g(y'_{x'}(k))$ so $f(x)''(k) < f(x')'(k)$ which means $f(x) \# f(x')$. Now suppose $y \in [0,1]$, then there exist $z_y \in [0,1]$ such that $z_y \equiv y$ and $z_y = z_y(0), z_y(1), z_y(2), \ldots$ with, for every $i \in \mathbb{N}$, $z'_y(i), z''_y(i) \in Q'$. Now consider $x = (g^{-1}(z'_y(0)), g^{-1}(z''_y(0))), (g^{-1}(z'_y(1)), g^{-1}(z''_y(1))), (g^{-1}(z'_y(2)), g^{-1}((z''_y(2))), \ldots$. Then

$f(x) = z_y$ so $f(x) \equiv y$, so $f(x) = y$.
Now we will prove $f(L_1) = L_2$. Suppose $x \in L_1$, then $x \in L_1'$ or $x \in L_1''$. Suppose $x \in L_1'$. First we note that $g(\{x \mid x \in Q \mid x \in L_1'\}) = \{x \mid x \in Q' | \ x \in L_2'\}$. Now, $x = \sum_{n=1}^{\infty} \frac{a_n}{6^n}$ such that $\forall n \in \mathbb{N}$, $a_n \in \{0, \frac{3^n+5}{2}\}$. Also $\lim_{m\to\infty} \sum_{n=1}^{m} \frac{a_n}{6^n} = x$. So, for all $k \in \mathbb{N}$ exists $m \in \mathbb{N}$ such that $|x - \sum_{n=1}^{m} \frac{a_n}{6^n}| \leq \frac{1}{2^k}$. Pick $l \in \mathbb{N}$ and find $k \in \mathbb{N}$ such that for all $x, y \in [0,1]$ if $|x - y| \leq \frac{1}{2^k}$ then $|f(x) - f(y)| \leq \frac{1}{2^l}$. Find $m \in \mathbb{N}$ such that $|x - \sum_{n=1}^{m} \frac{a_n}{6^n}| \leq \frac{1}{2^k}$. Then $|f(x) - f(\sum_{n=1}^{m} \frac{a_n}{6^n})| = |f(x) - g(\sum_{n=1}^{m} \frac{a_n}{6^n})| \leq \frac{1}{2^k}$. So $\lim_{m\to\infty} g(\sum_{n=1}^{m} \frac{a_n}{6^n}) = f(x)$, so $f(x) \in L_2'$. A similar argument proves if $x \in L_1''$ then $f(x) \in L_2''$. □

Lemma 5.4.8. *We can not prove L_2 is measurable.*

Proof. Suppose L_2 would be measurable. Then, as in the proof of lemma 5.4, L_2'' would be measurable and, since $L_2' = [0,1] \setminus L_2''$, also L_2' would be measurable. Suppose $\neg\exists n \in \mathbb{N}[n = k_1]$ then $L_2' = L_1'$ so $\mu(L_2') = \frac{1}{2}$. Suppose $\exists n \in \mathbb{N}[n = k_1]$ then $\mu(L_2') = 0$. This can be proven similar to the proof of lemma 5.4.3. Since $0 < \frac{1}{2}$ we know, either $\mu(L_2') > 0$ or $\mu(L_2') < \frac{1}{2}$. Suppose $\mu(L_2') > 0$ then $\neg\exists n \in \mathbb{N}[n = k_1]$. Suppose $\mu(L_2') < \frac{1}{2}$ then $\neg\neg\exists n \in \mathbb{N}[n = k_1]$. This means we can not prove that L_2 is measurable. □

Lemma 5.4.9. *(i) Every representative of E is not apart from $[0,1]$.*

(ii) Every representative of E does not deviate from $[0,1]$

(iii) Every representative of E does not coincide with $[0,1]$

Proof. By lemma 4.2 it is enough to show (i), (ii) and (iii) for L.

(i) We will prove $\neg[L \# [0,1]]$. Suppose there exists a $l \in L$ such that $l \# [0,1]$ or there exists an $x \in [0,1]$ such that $x \# L$. Then there exists an $x \in [0,1]$ such that $x \# L$, since $L \subseteq [0,1]$. So there exists an $x \in [0,1]$ such that for every $l \in L$ $[x \# l]$, which means for every $l \in L''$ $[x \# l]$, so $x \notin L''$ so $x \in L'$. Since also for every $l \in L''[x \# l]$ this will give $[x \# x]$, which is a contradiction.

(ii) We will prove $\neg[L \not\equiv [0,1]]$. So we have to proof $\neg\big[\exists l \in L \ \neg[l \in_0 [0,1]]\big]$ and $\neg\big[\exists x \in [0,1] \ \neg[x \in_0 L]\big]$. First we will prove $\neg\big[\exists l \in L \ \neg[l \in_0 [0,1]]\big]$. This is trivial since $L \subseteq [0,1]$, so $\forall l \in L \ [l \in_0 [0,1]]$. Now we will prove $\neg\big[\exists x \in [0,1] \ \neg[x \in_0 L]\big]$. Suppose $\big[\exists x \in [0,1] \ \neg[x \in_0 L]\big]$, find this x. Then $\neg(x \in_0 L)$, so $\neg(x \in_0 L'')$ so $\neg(x \in L'')$ thus $x \in L'$. But also $\neg(x \in_0 L'$, which is a contradiction.

(iii) This follows directly from 2.2.8.

□

Furthermore we can, for each representative E' of E find a totally bounded perfect set X such that for each $x \in X$ we can not prove $x \in E'$.

Lemma 5.4.10. *For every representative E' of E we can find a totally bounded perfect set X such that for each $x \in X$ we can not prove $x \in E'$.*

Proof. By lemma 4.10 we can prove our claim by proving it for L. Consider

$$X = \{\ x \mid x \in [0,1] \mid x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mid a_{k_1} = a_{k_1+1} = 1 \ \wedge \ a_i \in \{0,2\}\ \forall\ i \notin \{k_1, k_1+1\}\}$$

where k_1 is as in definition 1.3.1. We claim that X is a totally bounded perfect set and for each $x \in X$ we can not prove $x \in L$.
We will first show that X is totally bounded. Pick any $m \in \mathbb{N}$. Find the smallest $l \in \mathbb{N}$ such that $3^l \geq m$. We take $n = 2^l$. We consider three options:

1. $m \leq 3^{k_1-1}$

2. $3^{k_1-1} < m \leq 3^{k_1+1}$

3. $m > 3^{k_1+1}$

1. We define $p_0, p_1, \ldots, p_{n-1} \in X$ with $p_i = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ such that $a_j = 2 * i_j$ for each $j \leq l$ and $a_j = 0$ for each $j > l$ where $i_1, i_2, \ldots, i_l$ is the binary notation of i.

2. We define $p_0, p_1, \ldots, p_{2^{k_1-1}} \in X$ with $p_i = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ such that $a_j = 2 * i_j$ for each $j \leq k_1 - 1$ and $a_{k_1} = a_{k_1+1} = 1$ and $a_j = 0$ for each $j > k_1 - 1$ where $i_1, i_2, \ldots, i_{k_1-1}$ is the binary notation of i.

3. First define the smallest number $b \in \mathbb{N}$ such that $m \leq 3^{k_1+b+1}$. Now we define $p_0, p_1, \ldots, p_{2^{k_1-1+b}} \in X$ with $p_i = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ such that $a_j = 2 * i_j$ for each $j \leq k_1 - 1$ and $a_{k_1} = a_{k_1+1} = 1$ and $a_j = i'_{j-(k_1+1)}$ for each $j \leq k_1 + b + 1$ and $a_j = 0$ for each $j > k_1 + b + 1$ where $i_1, i_2, \ldots, i_{k_1-1}$ is the binary notation for $\lfloor \frac{i}{2^b} \rfloor$ and where $i'_1, i'_2, \ldots, i'_{k_1-1}$ is the binary notation for i modulo 2^b

We will now prove that X is closed. For this we need to show:

(i) $\forall x \in X\ \exists x' \in \bar{X}$ such that $x \equiv x'$

(ii) $\forall x' \in \bar{X}\ \exists x \in X$ such that $x \equiv x'$

Since $X \subseteq \bar{X}$ (i) is clear. Now take $x' \in \bar{X}$. We will construct $x = \sum_{n=1}^{\infty} \frac{b_n}{3^n} \in X$ by defining a_n for every $n \in \mathbb{N}$. Pick $n \in \mathbb{N}$ and find $y_n = \displaystyle\sum_{k=1}^{\infty} \frac{(a_n)_k}{3^k} \in X$ such that $|y_n - x'| < \frac{1}{3^{n+1}}$. Define $b_n = (a_n)_n$. We will show, for every $n \in \mathbb{N}$, $|x - x'| \leq \frac{1}{3^n}$, so by lemma 1.2.6 $x \equiv x'$. By lemma 1.1.7 it is sufficient to prove $|x - y_{n+1}| \leq \frac{1}{3^{n+1}}$ and $|y_{n+1} - x'| \leq \frac{1}{3^{n+2}}$. By definition of y_{n+1} we already know $|y_{n+1} - x'| \leq \frac{1}{3^{n+2}}$. Thus we will prove $|x - y_{n+1}| \leq \frac{1}{3^{n+1}}$. With the following claim we see it will be sufficient to prove $b_1 = (a_{n+1})_1, \ldots b_{n+1} = (a_{n+1})_{n+1}$.

Claim. *For* $y = \sum_{n=1}^{\infty} \frac{c_n}{3^n}, y' = \sum_{n=1}^{\infty} \frac{c'_n}{3^n} \in X$ *if* $c_1 = c'_1, \ldots, c_k = c'_k$ *then* $|y' - y| \leq \frac{1}{3^k}$ *and if* $|y' - y| < \frac{1}{3^k}$ *then* $c_1 = c'_1, \ldots, c_k = c'_k$.

Proof. Suppose $y = \sum_{n=1}^{\infty} \frac{c_n}{3^n}, y' = \sum_{n=1}^{\infty} \frac{c'_n}{3^n} \in X$. Suppose $c_1 = c'_1, \ldots, c_k = c'_k$ then $|y - y'| = |\sum_{n=k+1}^{\infty} \frac{c_n}{3^n} - \sum_{n=k+1}^{\infty} \frac{c'_n}{3^n}| \leq \frac{1}{3^k}$. Now suppose $|y' - y| < \frac{1}{3^k}$ and suppose there exists $m \leq k$ such that $c_m \neq c'_m$. Find the smallest $m \leq k$ such that $c_m \neq c'_m$. Decide $m < k_1, m \in \{k_1, k_1 + 1\}$ or $m > k_1 + 1$. Suppose $m < k_1$ then $c_m, c'_m \in \{0, 2\}$. Suppose without loss of generality $c_m = 0$ and $c'_m = 2$. Then $y \leq \sum_{n=1}^{m-1} \frac{c_n}{3^n} + \frac{1}{3^m}$ and $y' \geq \sum_{n=1}^{m-1} \frac{c_n}{3^n} + \frac{2}{3^m}$ so $|y - y'| \geq \frac{1}{3^m} \geq \frac{1}{3^k}$ which is a contradiction. Suppose $m \in \{k_1, k_1 + 1\}$, then $c_m = c'_m = 1$, which is a contradiction. Suppose $m > k_1 + 1$ then then $c_m, c'_m \in \{0, 2\}$. Suppose without loss of generality $c_m = 0$ and $c'_m = 2$. Then $y \leq \sum_{n=1}^{m-1} \frac{c_n}{3^n} + \frac{1}{3^m}$ and $y' \geq \sum_{n=1}^{m-1} \frac{c_n}{3^n} + \frac{2}{3^m}$ so $|y - y'| \geq \frac{1}{3^m} \geq \frac{1}{3^k}$ which is a contradiction. So not there exists $m \leq k$ such that $c_m \neq c'_m$ which means $c_m = c'_m$ for all $m \leq k$. □

We will prove $b_1 = (a_{n+1})_1, \ldots b_{n+1} = (a_{n+1})_{n+1}$ with induction, by using the above claim. By definition $b_1 = (a_1)_1$. Now suppose we know $b_1 = (a_n)_1, \ldots, b_n = (a_n)_n$. Then, since $|y_n - x'| \leq \frac{1}{3^{n+1}}$ and $|x' - y_{n+1}| \leq \frac{1}{3^{n+2}}$, by lemma 1.1.7, $|y_n - y_{n+1}| \leq \frac{1}{3^n}$. So, by the above claim, $(a_n)_1 = (a_{n+1})_1, \ldots (a_n)_n = (a_{n+1})_n$, so $b_1 = (a_{n+1})_1, \ldots b_{n+1} = (a_{n+1})_n$ and by definition $b_{n+1} = (a_{n+1})_{n+1}$. □

Moreover, Brouwer claims we can construct a representative E' of E for which we can define a set X of positive measure such that for each $x \in X$ we can not prove $x \in E'$. For this purpose Brouwer defines a set E_1, but we will show that we can not prove that E_1 is a representative of E.

We will define the set E_1 by first defining sets U_v and T_v for every $v \geq 2$, U_ω and T_ω and U_{k_1} and T_{k_1}. We note that, in defining these sets, Brouwers makes some mistakes. He defines the sets T_v, T_ω and T_{k_1} and then defines the sets U_v, U_ω and U_{k_1} as their complements. With the definition of T_v, T_ω and T_{k_1} it would not be clear what U_v, U_ω and U_{k_1} would be. Also, he claims that the sets T_v, T_ω and T_{k_1} are closed and the sets U_v, U_ω and U_{k_1} are regions, but it is not always the case that the complement of a closed set is a region. This is why we define the sets U_v, U_ω and U_{k_1} and then define T_v, T_ω and T_{k_1} as their complements.

Following the style of Brouwer, we first define, for every $v \geq 2$ a set U'_v. For this, we define a function, $\sigma : \mathbb{N}^{\geq 1} \to \mathbb{N}$ with $\sigma(v) = \frac{(v-1)(v+2)}{2}$. So $\sigma(1) = 0, \sigma(2) = 2, \sigma(3) = 5, \sigma(4) = 9$, etc. Also define, for every $v \geq 1$, $X_v := \{\frac{n}{3^{\sigma(v)}} | n < 3^{\sigma(v)}\}$. Now:

$$U'_v = \bigcup_{q \in X_v} \{x \mid x \in [0,1] \mid x = q + \big(\frac{1}{3^{\sigma(v)}} \sum_{n=1}^{\infty} \frac{a_n}{3^n}\big) \mid \exists m \in \mathbb{N} \,\big[\forall n < m[a_n \in \{0,2\}] \;\wedge\; a_m = 1 \;\wedge\; \forall n > m[a_n \in \{0,1,2\}] \;\wedge\; \exists\, p, q > m[a_p \neq 0 \;\wedge\; a_q \neq 2]\big] \;\}$$

So, for every $v \geq 2$, U'_v are all the $x \in [0,1]$ which we can write as $q + \big(\frac{1}{3^{\sigma(v)}} \sum_{n=1}^{\infty} \frac{a_n}{3^n}\big)$ and which are in the union of open intervals in the complement of small Cantor discontinua inside the intervals $[0, \frac{1}{3^{\sigma(v)}}], \dots, [\frac{3^{\sigma(v)}-1}{3^{\sigma v}}, 1]$.
Again, Brouwer seems to make the assumption that for every $x \in [0,1]$ we can find a sequence $a_1, a_2, a_3, \dots \in \{0,2\}$ such that $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$. With the above definition for U'_v it is not possible to prove that, for every $v \geq 2$, U'_v is a region. So we change the definition a little more. For this we define for every $M > 1$, similar as in the proof of 5.4.6, $E_1^M := \{x \in [0,1] \mid x = \sum_{n=1}^{M-1} \frac{a_n}{3^n} \mid \forall n \leq M-1[a_n \in \{0,2\}]\}$. Now define $U_v^M = \bigcup_{x \in E_1^M} \big(x + \frac{1}{3^{M-1}}, x + \frac{2}{3^M}\big)$ and $U := \bigcup_{M>1} U_v^M$. With this we define, for every $v \geq 2$,

$$U_v = \bigcup_{q \in X_v} \{x \in [0,1] \mid x = \frac{1}{3^{\sigma(v)}} y + q \mid y \in U\}$$

So U_v is really the union of all the open intervals in the complement of small Cantor discontinua inside the intervals $[0, \frac{1}{3^{\sigma(v)}}], \dots, [\frac{3^{\sigma(v)}-1}{3^{\sigma(v)}}, 1]$.
Since, for every $v \geq 2$, both U'_v and U_v give the same complement we choose to use U_v.

Lemma 5.4.11. *For every $v \geq 2$, $([0,1] \setminus U'_v) = ([0,1] \setminus U_v)$.*

Proof. We will not prove this is detail but refer to similar proofs.
As in the proof of lemma 5.4.1 we see that, for every $v \geq 2$, $([0,1] \setminus U'_v) = \bigcup_{q \in X_v} \{x \mid x \in [0,1] \mid x = q + \big(\frac{1}{3^{\sigma(v)}} \sum_{n=1}^{\infty} \frac{a_n}{3^n}\big) \mid \forall n \in \mathbb{N}[a_n \in \{0,2\}] \ \} = T'_v$. With a similar argument as in the proof of lemma 5.4.6 we can prove, for every $v \geq 2$:

(i) $T'_v \subseteq ([0,1] \setminus U_v)$

(ii) $([0,1] \setminus U_v) \subseteq T'_v$

This shows $([0,1] \setminus U'_v) = ([0,1] \setminus U_v)$. □

We now also define U_ω and U_{k_1}:

$$U_\omega = \bigcup_{v \geq 1} \Bigg(\bigcup_{q \in X_v} \Big(q + \frac{\sum_{n=0}^{v} 3^n}{3^{\sigma(v+1)}}, q + \frac{1 + \sum_{n=0}^{v} 3^n}{3^{\sigma(v+1)}}\Big)\Bigg)$$

$$U_{k_1} = \{ \ x \mid x \in [0,1] \mid \exists v \in \mathbb{N} \ [v = k_1 \wedge x \in U_v] \ \}$$

Furthermore, we define T_v, T_ω respectively T_{k_1} to be the complements of U_v, U_ω respectively U_{k_1}. For T_{k_1} this gives $T_{k_1} = \{ \ x \mid x \in [0,1] \mid \forall v \in \mathbb{N} \ [v = k_1 \rightarrow x \in T_v] \ \}$. Also, note the following: suppose $\exists v \in \mathbb{N} \ [v = k_1]$ then $T_{k_1} = T_v$ and $U_{k_1} = U_v$. Now suppose $\neg\exists v \in \mathbb{N} \ [v = k_1]$ then $T_{k_1} = [0,1]$ and $U_{k_1} = \emptyset$.
We will now prove a number of lemma's about the sets constructed above.

Lemma 5.4.12. *The sets U_v, U_w and U_{k_1} are regions.*

Proof. Clearly, for every $v \geq 2$, U_v is a region. Also, clearly U_ω is a region. Now define $\alpha = \alpha(0), \alpha(1), \alpha(2), \ldots$ with $\alpha(i) = (0, 0)$ if $i < k_1$ and $\alpha(i) = (\frac{k_1 - i}{3^{\sigma(k_1)}}, \frac{k_1 - i + 1}{3^{\sigma(k_1)}})$ for every $k_1 \leq i < k_1 + 3^{\sigma(k_1)}$ and $\alpha(i) = \alpha_{k_1}$ for all $i \geq k_1 + 3^{\sigma(k_1)}$. $U_{k_1} = \mathcal{R}(\alpha)$ so U_{k_1} is a region. □

Lemma 5.4.13. *For each $v \geq 2$, U_v is measurable and $\mu(U_v) = 1$.*

Proof. Pick $v \geq 2$. Then the number of elements in X_v is $3^{\sigma(v)}$. We have $\mu(U_v) = \mu(\bigcup_{q \in X_v} (q, q + \frac{1}{3^{\sigma(v)}})) = \displaystyle\sum_{n=0}^{3^{\sigma(v)}-1} \frac{1}{3^{\sigma(v)}} = 1$. □

Lemma 5.4.14. $\mu(T_\omega) \geq \frac{5}{6}$.

Proof. We will prove $\mu(U_\omega) \leq \frac{1}{6}$. U_ω is a measurable region since, for every $m \in \mathbb{N}$ there exists $v \in \mathbb{N}$ such that for all $n \geq v$, $\mu(\displaystyle\bigcup_{q \in X_n} (q + \frac{\sum_{k=0}^{n} 3^k}{3^{\sigma(n+1)}}, q + \frac{1 + \sum_{k=0}^{n} 3^k}{3^{\sigma(n+1)}})) \leq \frac{1}{m}$. Now $\mu(U_\omega) = \mu(\bigcup_{v \geq 1} \left(\bigcup_{q \in X_v} \left(q + \frac{\sum_{n=0}^{v} 3^n}{3^{\sigma(v+1)}}, q + \frac{1 + \sum_{n=0}^{v} 3^n}{3^{\sigma(v+1)}} \right) \right)) \leq \displaystyle\sum_{n=1}^{\infty} \frac{3^{\sigma(v)}}{3^{\sigma(v+1}} = \sum_{n=1}^{\infty} 3^{\sigma(v) - \sigma(v+1)} = \sum_{n=1}^{\infty} \frac{1}{3^{v+1}} = \frac{1}{6}$. □

Here the first inequality again holds since, for every $v \geq 1$ the number of elements of X_v is $3^{\sigma(v)}$.
We define the set E_1 as the union of T_{k_1} and U_{k_1}, so $E_1 = T_{k_1} \cup U_{k_1}$. We define X as the intersection of $T_\omega, U_2, U_3, \ldots$, so $X = T_\omega \cap (\bigcap_{v=2}^{\infty} U_v)$.

Lemma 5.4.15. *X has a positive measure (i.e. there exists an $n \in \mathbb{N}$ with $|\mu(X) - 0| > \frac{1}{n}$) and for every $x \in X$ we can not prove $x \in E_1$.*

Proof. Since for every $v \geq 2$, $\mu(U_v) = 1$ we have $\mu(\bigcap_{v=2}^{\infty} U_v) = 1$. This means $\mu(X) \geq \frac{5}{6}$. Also, suppose $x \in X$ and suppose we prove $x \in E_1$, then we prove $x \in U_{k_1}$ or we prove $x \in T_{k_1}$.

- Suppose we prove $x \in T_{k_1}$ and also suppose $\exists v \in \mathbb{N}\ [v = k_1]$. Find $v \in \mathbb{N}$ such that $v = k_1$, then $x \in T_v$. By definition of T_v, $x \notin U_v$. But then $x \notin X$. So we have $\neg\exists v \in \mathbb{N}\ [v = k_1]$. We can not prove this, so we can not prove $x \in T_{k_1}$.

- Suppose $x \in U_{k_1}$ then $U_{k_1} \neq \emptyset$ so $\neg\neg\exists v \in \mathbb{N}\ [v = k_1]$. We can not prove this, so we can not prove $x \in U_{k_1}$.

So, for every $x \in X$ we can not prove $x \in E_1$. □

Now we will show that we can not prove that E_1 is a representative of E.

Lemma 5.4.16. *We can not prove $L \sim E_1$.*

Proof. Suppose we prove $L \sim E_1$, then there would exists a uniformly continuous bijection $f : [0, 1] \to [0, 1]$ such that $f(L) = E_1$ and such that it's inverse f^{-1} is uniformly continuous as well. Now suppose $\neg\exists v \in \mathbb{N}\ [v = k_1]$, then $E_1 = [0, 1]$. But then $L = f^{-1}(E_1) = f^{-1}([0, 1]) = [0, 1]$. So $\neg\neg\exists v \in \mathbb{N}\ [v = k_1]$. So we can not prove such an f exists. □

Lastly, we will show, for every representative E' of E there exists discontinuous functions $f : E' \to \mathbb{R}$.

Lemma 5.4.17. *For every representative E' of E there exists discontinuous functions $f : E' \to \mathbb{R}$.*

Proof. By lemma 4.3 it is enough to prove this for L. Now define $f : L \to \mathbb{R}$ with $f(x) = 0$ if $x \in L'$ and $f(x) = 1$ if $x \in L''$. To prove f is discontinuous we consider any $x \in L'$ and $n = 2$. Since $x \in L'$ we can find a sequence $a_1, a_2, a_3, \dots \in \{0, 2\}$ such that $x = \sum_{n=1}^{infty} \frac{a_n}{3^n}$. Pick any $m \in \mathbb{N}$ and find $k \in \mathbb{N}$ such that $\sum_{n=k+1}^{\infty} \frac{2}{3^n} \leq \frac{1}{2^m}$. Define $y = \sum_{n=1}^{\infty} \frac{b_n}{3^n}$ with $b_n = a_n$ for all $n \leq k$ and $b_{k+n} = 1$ for all $n \geq 1$. Then $y \in L''$. We have $|x - y| \leq \sum_{n=k+1}^{\infty} \frac{2}{3^n} \leq \frac{1}{2^m}$ but $|f(x) - f(y)| = 1 > \frac{1}{4}$. □

5.5 Example 6

We define F to be the geometric type of M (10), where $M = M' \cup M''$ and:

$$M' = \{\ x \mid x \in [0,1] \mid x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mid \exists m \in \mathbb{N} \forall n > m [a_n \neq 1]\}$$

$$M'' = \{\ x \mid x \in [0,1] \mid x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mid \forall m \in \mathbb{N} \exists n \in \mathbb{N} [a_n = 1]\}$$

Lemma 5.5.1. *$M' \subseteq ([0,1] \setminus M'')$ and $M'' \subseteq ([0,1] \setminus M'$ so $M' \cap M'' = \emptyset$.*

Proof. Suppose $x \in M'$ then $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ and we can find $m \in \mathbb{N}$ such that for all $n > m$, $a_n \neq 1$. Suppose $x \in M''$ then for all $k \in \mathbb{N}$ there exists $n > k$ such that $a_n = 1$. Now find $n > m$ such that $a_n = 1$. This is a contradiction so $x \notin M''$.
Now suppose $x \in M''$ and suppose $x \in M'$, then by the above we get a contradiction. So $x \notin M'$. □

Corollary 5.5.2. *For every representative F' of F we have F' is of the form $X' \cup X''$ such that $X' \cap X'' = \emptyset$.*

Proof. Suppose F' is a representative of F, then there exists a uniformly continuous bijection $f : [0,1] \to [0,1]$ such that $f(M) = F'$ and such that f^{-1} is uniformly continuous. But $f(M) = f(M' \cup M'') = f(M') \cup f(M'') = F'$. So define $X' = f(M')$ and $X'' = f(M'')$. Now suppose there exists $x \in (X' \cap X'')$ then $x \in X'$ so $x \in f(M')$ which means $f^{-1}(x) \in M'$. This means $f^{-1}(x) \notin M''$ so $x \notin X''$. This is a contradiction so $X = \emptyset$. □

Lemma 5.5.3. *1. Every representative of F is not apart from $[0,1]$.*

2. Every representative of F does not coincide with $[0,1]$.

Proof. By lemma 4.2 it is enough to show $(i), (ii)$ and (iii) for M.

(10) This is K from example 6 of Brouwers article.

(i) Suppose $M \# [0,1]$. Then, there exists $x \in [0,1]$ such that $[x \# M]$ or there exists $x \in M$ such that $[x \# [0,1]]$. Since $M \subseteq [0,1]$ we know there exists $x \in [0,1]$ such that $[x \# M]$. Find $x \in [0,1]$ such that $x \# M$. Then $x \# M'$. We define, for all $k \in \mathbb{N}$, $M'_k := \Big\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mid \sum_{n=1}^{\infty} \frac{a_n}{3^n} \in M' \mid \exists l \leq k \Big[\exists n_1 \neq \cdots \neq n_l \big[\forall i \leq l [a_{n_i} = 1] \;\wedge\; \forall n \notin \{n_1, \ldots, n_l\}[a_n \neq 1] \big] \Big] \Big\}$. So M'_k is the set which consist of all $x \in M'$ that contain at most k terms where the nominator equals 1. Then $M' = \bigcup_{k \in \mathbb{N}} M'_k$. Now, for every $k \in \mathbb{N}$, M'_k is a fan and $\forall z \in M'_k \exists n \in \mathbb{N}[x'(n) > z \;\vee\; x''(n) < z]$. So, by the fan theorem, for every $k \in \mathbb{N} \exists N_k \forall z \in M'_k \exists n \leq N_k [x'(n) > z \;\vee\; x''(n) < z]$. We will prove there exists $w \in M''$ with $x \equiv w$. Since $[x \# M'']$ this is a contradiction, so $\neg[M \# [0,1]]$. For this, we will prove the following:

(1) There exists a sequence $a_1, a_2, a_3, \dots \in \{0,1,2\}$ such that $\lim_{k \to \infty} \sum_{n=1}^{k} \frac{a_n}{3^n} \equiv x$.

(2) $\sum_{n=1}^{\infty} \frac{a_n}{3^n} \in M''$.

(1) Define $(a_n)_{n=1}^{\infty}$ with induction and such that, for every $k \in \mathbb{N}$, $\sum_{n=1}^{k} \frac{a_n}{3^n} < x < \sum_{n=1}^{k} \frac{a_n}{3^n} + \frac{1}{3^k}$. For $k = 1$, decide $x < \frac{1}{3}$ or $x > \frac{1}{3}$. This is possible since $x \# M'$. If $x < \frac{1}{3}$ then $a_1 = 0$. If $x > \frac{1}{3}$ then decide $x < \frac{2}{3}$ or $x > \frac{2}{3}$. If $x < \frac{2}{3}$ then $a_1 = 1$. If $x > \frac{2}{3}$ then $a_1 = 2$. This means $\sum_{n=1}^{1} \frac{a_n}{3^n} < x < \sum_{n=1}^{1} \frac{a_n}{3^n} + \frac{1}{3}$. Now suppose we defined $a_1, a_2, \dots, a_k$. Then decide $x < \sum_{n=1}^{k} \frac{a_n}{3^n} + \frac{1}{3^{k+1}}$ of $x > \sum_{n=1}^{k} \frac{a_n}{3^n} + \frac{1}{3^k}$. If $x < \sum_{n=1}^{k} \frac{a_n}{3^n} + \frac{1}{3^{k+1}}$ then $a_{k+1} = 0$, else decide $x < \sum_{n=1}^{k} \frac{a_n}{3^n} + \frac{2}{3^{k+1}}$ or $x > \sum_{n=1}^{k} \frac{a_n}{3^n} + \frac{2}{3^{k+1}}$. If $x < \sum_{n=1}^{k} \frac{a_n}{3^n} + \frac{2}{3^{k+1}}$, then $a_{k+1} = 1$, else $a_{k+1} = 2$. Clearly, for every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $|\sum_{n=0}^{\infty} - x| \leq \frac{1}{n}$. So, by lemma 1.2.6, $\lim_{k \to \infty} \sum_{n=1}^{k} \frac{a_n}{3^n} = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \equiv x$.

(2) To show $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ we have to define a real number $y = y(0), y(1), y(2), \dots$ with $y'(k) = \sum_{n=1}^{k+1} \frac{a_n}{3^n} - \frac{1}{3^k}$ and $y''(k) = \sum_{n=1}^{k+1} \frac{a_n}{3^n} + \frac{1}{3^k}$. With y we will prove for all $m \in \mathbb{N}$ there exists $n > m$ such that $a_n = 1$.

Pick $k \in \mathbb{N}$, then $x'(k) < x < \sum_{n=1}^{k} \frac{a_n}{3^n} + \frac{1}{3^k} < y''(k)$ and $y'(k) = \sum_{n=1}^{k+1} \frac{a_n}{3^n} - \frac{1}{3^k} \leq x < x''(k)$, so $x \equiv y$. We will now prove, with induction, $\sum_{n=1}^{\infty} \frac{a_n}{3^n} \in M''$.

Since $y \equiv x$ we have, for every $k \in \mathbb{N} \exists N_k \forall z \in M'_k \exists n \leq N_k [x'(n) > z \;\vee\; x''(n) < z]$.

Find N_0. Consider $\sum_{n=1}^{N_0} \frac{a_n}{3^n}$ and suppose for all $n \le N_0$, $a_n \in \{0, 2\}$. Then $\sum_{n=1}^{N_0} \frac{a_n}{3^n} \in M'_0$, so $y''(N_0) < \sum_{n=1}^{N_0} \frac{a_n}{3^n}$ or $y'(N_0) > \sum_{n=1}^{N_0} \frac{a_n}{3^n}$. But $y''(N_0) = \sum_{n=1}^{N_0+1} \frac{a_n}{3^n} + \frac{1}{3^{N_0}} \ge \sum_{n=1}^{N_0} \frac{a_n}{3^n}$ and $y'(N_0) = \sum_{n=1}^{N_0+1} \frac{a_n}{3^n} - \frac{1}{3^{N_0}} < \sum_{n=1}^{N_0+1} \frac{a_n}{3^n} - \frac{2}{3^{N_0+1}} \le \sum_{n=1}^{N_0+1} \frac{a_n}{3^n}$. This means $\exists n \le N_0$ such that $a_n = 1$. Now suppose we have proven $\exists n_1 \ne \dots \ne n_k \le \max(N_0, \dots, N_k)$ such that $a_{n_i} = 1$ for all $i \le k$ and $a_n \ne 1$ for all $n \notin \{n_1, \dots, n_k\}$. Then find N_{k+1} and $m = \max(N_0, \dots, N_{k+1})$. Consider $\sum_{n=1}^{m} \frac{a_n}{3^n}$ and suppose $\exists n_1 \ne \dots \ne n_k \le m$ such that $a_{n_i} = 1$ for all $i \le k$ and $a_n \ne 1$ for all $n \notin \{n_1, \dots, n_k\}$. Then $\sum_{n=1}^{m} \frac{a_n}{3^n} \in M'_k$. So $y''(N_k) < \sum_{n=1}^{m} \frac{a_n}{3^n}$ or $y'(N_k) > \sum_{n=1}^{N_0} \frac{a_n}{3^n}$. But $y''(N_k) = \sum_{n=1}^{N_k+1} \frac{a_n}{3^n} + \frac{1}{3^{N_k}} \ge \sum_{n=1}^{N_k} \frac{a_n}{3^n} + \sum_{n=N_k+2}^{\infty} \frac{2}{3^n} \ge \sum_{n=1}^{m} \frac{a_n}{3^n}$ and $y'(N_k) = \sum_{n=1}^{N_k+1} \frac{a_n}{3^n} - \frac{1}{3^{N_k}} < \sum_{n=1}^{m} \frac{a_n}{3^n} - \frac{2}{3^{N_0+1}} \le \sum_{n=1}^{m} \frac{a_n}{3^n}$. This means $\exists n_1 \ne \dots \ne n_{k+1} \le m$ such that $a_{n_i} = 1$ for all $i \le k+1$. With induction, we see $\sum_{n=1}^{\infty} \frac{a_n}{3^n} \in M''$.

(ii) This follows directly from lemma 2.2.8.

□

Lemma 5.5.4. *M is measurable and $\mu(M) = 1$.*

Proof. We will prove M'' is measurable and $\mu(M'') = 1$, then also M is measurable and $\mu(M) = 1$.
To prove M'' is measurable we have to proof $\chi_{M''}$ is measurable. We will define a set $Y_M := \left\{x \mid x \in [0,1] \mid \exists l \in \mathbb{N}\big[x = \sum_{n=1}^{l} \frac{a_n}{3^n} \ \wedge\ \forall n \le l[a_n \in \{0,1,2\}]\big]\right\}$. Since for every $x \in Y_M$, $x \in \mathbb{Q}$ we can enumerate Y_M with $q_0, q_1, q_2, \dots$.
Now define an infinite sequence of measurable regions $X_0, X_1, X_2, \dots$ such that, for each $n \in \mathbb{N}$, $X_n = \mathcal{R}(\alpha_n(0), \alpha_n(1), \alpha_n(2), \dots)$ with $\alpha_n(i) = (q_i - \frac{1}{2^{n+i+2}}, q_i + \frac{1}{2^{n+i+2}})$ for every $i \in \mathbb{N}$. Then $\mu(X_n) \le \frac{1}{2^n}$ for every $n \in \mathbb{N}$. Pick $x \in [0,1]$ and suppose $x \notin X_n$ then $\forall m \in \mathbb{N} \forall k \in \mathbb{N}[q_m - \frac{1}{2^{n+m+2}} \ge x'(k) \ \vee\ x''(k) \ge q_m + \frac{1}{2^{n+m+2}}]$. Similar to the proof of lemma 5.5.3 we can prove $x \in M''$.
Now define, for every $n \in \mathbb{N}$, $(v_n)_0, \dots, (v_n)_n$ with $((v_n)_i)_0 = (\frac{i}{n+1}, \frac{i+1}{n+1})$ and $((v_n)_i)_1 = (1 - \frac{1}{2^n}, 1)$. Then $\mathrm{Ar}^* = \frac{n+1}{2^n(n+1)} = \frac{1}{2^n}$.
Suppose $x \in [0,1]$ and suppose $x \notin X_n$, then by the above $x \in M''$ which means $\chi_{M''}(x) = 1$. Also $1 - \frac{1}{2^n} \le 1 \le 1$ so $\chi_{M''}(x) \in ((v_n)_i)_1$. This gives $\mu(M'') = \lim_{n\to\infty} \frac{n+1}{n+1} = 1$. □

Corollary 5.5.5. *M' is measurable and $\mu(M') = 0$.*

Proof. Since M is measurable and $M = M' \cup M''$ such that $M' \cap M'' = \emptyset$ we know, by lemma 3.3.12, both M' and M'' are measurable. Also, since $\mu(M'') = 1$ we have $\mu([0,1] \setminus M'') = 0$. Since $M' \subseteq ([0,1] \setminus M'')$ we know $\mu(M') = 0$. □

Lemma 5.5.6. *Every measurable representative of F has measure 1.*

Proof. Suppose F' is a representative of F and suppose F' is measurable. Suppose $\mu(F') < 1$. This means $\mu([0,1] \setminus F') > 0$ and thus, by lemma 3.3.11, we know $\exists x \in ([0,1] \setminus F')$ such that $x \# F'$. By lemma 5.5.3 this is not possible, so $\mu(F') = 1$. □

So every measurable representative of F has measure 1, but there exist representatives of F for which we can not prove they are measurable. We will show this with the set M_2. To define M_2 we first have to define a measurable representative of F, M_1. Brouwer did not mention these representatives, but we give them to show that we can not prove that every representative of F is measurable.

We define $M_1 = M_1' \cup M_1''$ where $M_1' = L_1' \cup M'''$ and:

$$M''' := \Big\{ \ x \mid x \in [0,1] \mid x = \sum_{n=1}^{k} \frac{a_n}{6^n} \ + \frac{2}{6^k} \sum_{m=1}^{\infty} \frac{b_m}{3^m} \\ \mid \forall n < k[a_k \in \{0, \frac{3^n+5}{2}\}] \ \wedge \ a_k = \frac{3^k+1}{2} \ \wedge \\ \exists l \in \mathbb{N} \forall m > l[b_m \neq 1] \ \Big\}$$

$$M_1'' = \Big\{ \ x \mid x \in [0,1] \mid x = \sum_{n=1}^{k} \frac{a_n}{6^n} \ + \frac{2}{6^k} \sum_{m=1}^{\infty} \frac{b_m}{3^m} \\ \mid \forall n < k[a_k \in \{0, \frac{3^n+5}{2}\}] \ \wedge \ a_k = \frac{3^k+1}{2} \ \wedge \\ \forall l \in \mathbb{N} \exists m > l[b_m = 1] \Big\}$$

We will first prove that M_1 is a representative of F and then prove that M_1 is measurable.

Lemma 5.5.7. $M \sim M_1$

Proof. This is very similar to the proof of 5.4.5. We will define a total bijection $f : [0,1] \to [0,1]$. For this we define two sets $Q, Q' \subseteq [0,1]$ with:

$$Q := \{x \in [0,1] \mid \exists m \in \mathbb{N}\big[x = \sum_{n=1}^{m} \frac{a_n}{3^n} \ \wedge \ \forall n \leq m[a_n \in \{0,1,2\}] \ \big]\}$$

$$Q' := \{x \in [0,1] \mid \exists m \in \mathbb{N}\big[x = \sum_{n=1}^{m} \frac{a_n}{6^n} \ \wedge \ \forall n \leq m[a_n \in \{0, \frac{3^n+5}{2}\}] \ \big]\} \cup \\ \{x \in [0,1] \mid \exists k, l \in \mathbb{N}\big[x = \sum_{n=1}^{k} \frac{a_n}{6^n} + \frac{2}{6^k} \sum_{m=1}^{l-k} \frac{b_m}{3^m} \ \wedge \\ \forall n < k[a_n \ \in \{0, \frac{3^n+5}{2}\}] \ \wedge \ a_k = \frac{3^k+1}{2} \ \wedge \\ \forall m \leq l-k[a_m \in \{0,1,2\}] \ \big]\}$$

So Q are all the elements from M which we can write as a finite sum and Q' are all the elements from M_1 which we can write as a finite sum. We define a bijection $g : Q \to Q'$ and use this bijection to define f.

For every $q = \sum_{n=1}^{l} \frac{a_n}{3^n} \in Q$, define:

$$g(q) = \begin{cases} \sum_{n=1}^{l} \frac{b_n}{6^n} & \text{if } \forall n \leq l[a_n \neq 1] \\ \sum_{n=1}^{k} \frac{b_n}{6^n} + \frac{2}{6^k} \sum_{m=1}^{l-k} \frac{c_m}{3^m} & \text{if } \exists k \leq l[a_k = 1] \end{cases}$$

with, if $\forall n \leq l[a_n \neq 1]$ then, for all $n \leq l$, $b_n = 0$ if $a_n = 0$ and $b_n = \frac{3^n+5}{2}$ if $a_n = 2$ and if $\exists k \leq l[a_k = 1]$ then, for all $n < k$, $b_n = 0$ if $a_n = 0$ and $b_n = \frac{3^n+5}{2}$ if $a_n = 2$, $b_k = \frac{3^k+1}{2}$ and $\forall m \leq l-k$, $c_m = a_{m+k}$.

We will prove g is a bijection. Take $q = \sum_{n=1}^{l} \frac{a_n}{3^n}, q' = \sum_{n=1}^{l'} \frac{a'_n}{3^n} \in Q$ and decide $l \geq l'$ or $l' < l$. Suppose, without loss of generality, $l \geq l'$. Then define $q' = \sum_{n=1}^{l} \frac{a'_n}{3^n}$ with $a'_n = 0$ for all $l' < n \leq l$. Suppose $q \neq q'$. Find the smallest $k \leq l$ such that $a_k \neq a'_k$. Suppose $a_k = 0$ and $a'_k = 2$. Then $g(q) < {}_{n=1}^{k-1} \frac{a_n}{6^n} + \frac{\frac{3^k+1}{2}}{6^k}$ and $g(q') \geq {}_{n=1}^{k-1} \frac{a_n}{6^n} + \frac{\frac{3^k+5}{2}}{6^k}$. Suppose $a_k = 1$ and $a'_k = 2$. Then $g(q) < \sum_{n=1} k - 1 \frac{a_n}{6^n} + \frac{\frac{3^k+1}{2}}{6^k} + \frac{2}{6^k}$ and $g(q') \geq g(q) < \sum_{n=1} k - 1 \frac{a_n}{6^n} + \frac{\frac{3^k+5}{2}}{6^k}$. Suppose $a_k = 1$ and $a'_k = 0$. Then $g(q) \geq \sum_{n=1}^{k-1} \frac{a_n}{6^n} + \frac{\frac{3^k+1}{2}}{6^k}$ and $g(q') < \sum_{n=1}^{k-1} \frac{a_n}{6^n} + \frac{\frac{3^k+1}{2}}{6^k}$.

Now we can define f.

For every $x \in [0,1]$ there exists $y_x \in [0,1]$ such that $y_x \equiv x$ and $y_x = y_x(0), y_x(1), y_x(2), \ldots$ with, for every $i \in \mathbb{N}$, $y'_x(i), y''_x(i) \in Q$. Pick $x \in [0,1]$ and find y_x. Define $f(x) = (g(y'_x(0)), g(y''_x(0))), (g(y'_x(1)), g(y''_x(1$
Clearly, f is a total function. This means, by the uniform continuity theorem, f is uniformly continuous. We will prove f is a bijection. Also, we will prove $f(M) = M_1$. This will prove $M_1 \sim M$.

First we will prove f is a bijection. Suppose $x, x' \in [0,1]$ and $x \# x'$. Find $y_x, y_{x'}$. Then $y_x \# y_{x'}$ so there exists $k \in \mathbb{N}$ such that $y''_x(k) < y'_{x'}(k)$ or $y''_{x'}(k) < y'_x(k)$. Suppose, without loss of generality, $y''_x(k) < y'_{x'}(k)$ then $g(y''_x(k)) < g(y'_{x'}(k))$ so $f(x)''(k) < f(x')'(k)$ which means $f(x) \# f(x')$. Now suppose $y \in [0,1]$, then there exist $z_y \in [0,1]$ such that $z_y \equiv y$ and $z_y = z_y(0), z_y(1), z_y(2), \ldots$ with, for every $i \in \mathbb{N}$, $z'_y(i), z''_y(i) \in Q'$. Now consider $x = (g^{-1}(z'_y(0)), g^{-1}(z''_y(0))), (g^{-1}(z'_y(1)), g^{-1}(z''_y(1))), (g^{-1}(z'_y(2)), g^{-1}((z''_y(2))), \ldots$. Then $f(x) = z_y$ so $f(x) \equiv y$, so $f(x) = y$.

Now we will prove $f(M) = M_1$. Suppose $x \in M$, then $x \in M'$ or $x \in M''$. Suppose $x \in M'$ and first suppose $x \in L'$. Then $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ with $\forall n \in \mathbb{N}[a_n \neq 1]$. Also $\lim_{m \to \infty} \sum_{n=1}^{m} \frac{a_n}{3^n} = x$. So, for all $k \in \mathbb{N}$ exists $m \in \mathbb{N}$ such that $|x - \sum_{n=1}^{m} \frac{a_n}{3^n}| \leq \frac{1}{2^k}$. Pick $l \in \mathbb{N}$ and find $k \in \mathbb{N}$ such that for all $x, y \in [0,1]$ if $|x - y| \leq \frac{1}{2^k}$ then $|f(x) - f(y)| \leq \frac{1}{2^l}$. Find $m \in \mathbb{N}$ such that $|x - \sum_{n=1}^{m} \frac{a_n}{3^n}| \leq \frac{1}{2^k}$. Then $|f(x) - f(\sum_{n=1}^{m} \frac{a_n}{3^n})| = |f(x) - g(\sum_{n=1}^{m} \frac{a_n}{3^n})| \leq \frac{1}{2^k}$. So $\lim_{m \to \infty} g(\sum_{n=1}^{m} \frac{a_n}{3^n}) = \lim_{m \to \infty} \sum_{n=1}^{m} \frac{b_n}{6^n} = f(x)$.

Since, for all $n \in \mathbb{N}$, $a_n \neq 1$ we have, for all $m \in \mathbb{N}$ for all $n \leq m$, $b_n \neq \frac{3^n+1}{2}$. So $f(x) \in L_1'$.
Now suppose $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \in M'$ and there exists $m \in \mathbb{N}$ such that $a_m = 1$. Also, find $p \in \mathbb{N}$ such that for all $n > p$, $a_n \neq 1$. Now, with the same argument as above, but starting at m, we get $\lim_{k\to\infty} g(\sum_{n=1}^{k} \frac{a_n}{3^n}) = \lim_{k\to\infty} \sum_{n=1}^{m} \frac{b_n}{6^n} + \frac{2}{6^m} \sum_{l=1}^{k-m} \frac{c_l}{3^l} = f(x)$. Pick $k \in \mathbb{N}$. Since $a_m = 1$ we have $b_m = \frac{3^m+1}{2}$ and for all $l \leq k - m$ we have $c_m = a_{m+k}$. So, for all $m \geq k - p$, $c_m \neq 1$. This means $f(x) \in M_1'$.
A similar argument proves if $x \in M''$ then $f(x) \in M_1''$. □

Lemma 5.5.8. *M_1' is measurable and $\mu(M_1') = \frac{1}{2}$.*

Proof. We first define $Q_0 = \{0\}$ and, for each $k \geq 1$, $Q_k = \{x \mid x \in L_1' \mid x = \sum_{n=1}^{k} \frac{a_n}{3^n}\}$. Now define $Q = \bigcup_{k\in\mathbb{N}} Q_k$. So all the elements from Q are all the elements from L_1' which we can write like a finite sum. Enumerate, for all $k \in \mathbb{N}$, Q_k with $q_1^k, \dots, q_{2^k}^k, \dots$. Pick $k \geq 2$ and $i \leq 2^{k-1}$ and consider $q_i^{k-1} = \sum_{n=1}^{k-1} \frac{a_n}{6^n}$. Then consider $(q_i^{k-1} + \frac{\frac{3^k+1}{2}}{6^k}, q_i^{k-1} + \frac{\frac{3^k+5}{2}}{6^k})$. In M_1' we actually make a 'small version' of M' in every interval of the form $(q_i^{k-1} + \frac{\frac{3^k+1}{2}}{6^k}, q_i^{k-1} + \frac{\frac{3^k+5}{2}}{6^k})$ for some $q_i^{k-1} \in Q$. All of these 'small versions' have measure 0 so M_1' has measure 0.
We will now define this 'small version' of M' in $(q_i^{k-1} + \frac{\frac{3^k+1}{2}}{6^k}, q_i^{k-1} + \frac{\frac{3^k+5}{2}}{6^k})$ with $W_{q_i^{k-1}}$. So $W_{q_i^{k-1}} = \{\ x \mid x \in M_1' \mid x = q_i^{k-1} + \frac{\frac{3^k+1}{2}}{6^k} + \frac{2}{6^k} \sum_{m=1}^{\infty} \frac{b_m}{3^m} \mid \exists l \in \mathbb{N} \forall m > l[b_m \neq 1]\}$. Then $W_{q_i^{k-1}} \subseteq (q_i^{k-1} + \frac{\frac{3^k+1}{2}}{6^k}, q_i^{k-1} + \frac{\frac{3^k+5}{2}}{6^k}) \subseteq [0, 1]$.
Furthermore $\mu(M') = \mu(\{\ x \mid x \in [0, 1] \mid x = \sum_{m=1}^{\infty} \frac{b_m}{3^m} \mid \exists l \in \mathbb{N} \forall m > l[b_m \neq l]\}) = 0$, so, by lemma 3.3.13 $\mu(\{\ x \mid x \in [0, 1] \mid x = \frac{2}{6^k} \sum_{m=1}^{\infty} \frac{b_m}{3^m} \mid \exists l \in \mathbb{N} \forall m > l[b_m \neq l]\}) = 0$ and $\mu(W_{q_i}) = 0$.
Also $M''' = \bigcup_{k\geq 1}(\bigcup_{i\leq 2^{k-1}} W_{q_i^{k-1}})$ so $\mu(M''') = \sum_{k=1}^{\infty}(\sum_{i=1}^{2^{k-1}} \mu(W_{q_i}^{k-1}) = \sum_{k=1}^{\infty} 0 = 0.$. Since M_1' is a union of two disjoint sets we have $\mu(M_1') = \mu(L_1') + \mu(M''') = \frac{1}{2}$. □

Lemma 5.5.9. *M_1'' is measurable and $\mu(M_1'') = \frac{1}{2}$.*

Proof. We first define $Q_0 = \{0\}$ and, for each $k \geq 1$, $Q_k = \{x \mid x \in L_1' \mid x = \sum_{n=1}^{k} \frac{a_n}{3^n}\}$. Now define $Q = \bigcup_{k\in\mathbb{N}} Q_k$. So all the elements from Q are all the elements from L_1' which we can write like a finite sum. Enumerate, for all $k \in \mathbb{N}$, Q_k with $q_1^k, \dots, q_{2^k}^k, \dots$. Pick $k \geq 2$ and $i \leq 2^{k-1}$ and consider $q_i^{k-1} = \sum_{n=1}^{k-1} \frac{a_n}{6^n}$. Then consider $(q_i^{k-1} + \frac{\frac{3^k+1}{2}}{6^k}, q_i^{k-1} + \frac{\frac{3^k+5}{2}}{6^k})$. In M_1'' we actually make a 'small version' of M'' in every interval of the form $(q_i^{k-1} + \frac{\frac{3^k+1}{2}}{6^k}, q_i^{k-1} + \frac{\frac{3^k+5}{2}}{6^k})$ for some $q_i^{k-1} \in Q$. All of these 'small versions' have measure $\frac{2}{6^k}$.

We will now define this 'small version' of M'' in $(q_i^{k-1} + \frac{\frac{3^k+1}{2}}{6^k}, q_i^{k-1} + \frac{\frac{3^k+5}{2}}{6^k})$ with $W_{q_i^{k-1}}$. So $W_{q_i^{k-1}} = \{\ x \mid x \in M_1' \mid x = q_i^{k-1} + \frac{\frac{3^k+1}{2}}{6^k} + \frac{2}{6^k}\sum_{m=1}^{\infty} \frac{b_m}{3^m} \mid \forall l \in \mathbb{N} \exists m > l[b_m = 1]\}$. Then $W_{q_i^{k-1}} \subseteq (q_i^{k-1} + \frac{\frac{3^k+1}{2}}{6^k}, q_i^{k-1} + \frac{\frac{3^k+5}{2}}{6^k}) \subseteq [0,1]$.

Furthermore $\mu(M'') = \mu(\{\ x \mid x \in [0,1] \mid x = \sum_{m=1}^{\infty} \frac{b_m}{3^m} \mid \forall l \in \mathbb{N} \exists m > l[b_m = l]\}) = 1$, so by lemma 3.3.13 $\mu(\{\ x \mid x \in [0,1] \mid x = \frac{2}{6^k}\sum_{m=1}^{\infty} \frac{b_m}{3^m} \mid \exists l \in \mathbb{N} \forall m > l[b_m \neq l]\}) = \frac{2}{6^k}$ and $\mu(W_{q_i^{k-1}}) = \frac{2}{6^k}$.

Also $M_1'' = \bigcup_{k\geq 1}(\bigcup_{i \leq 2^{k-1}} W_{q_i^{k-1}})$ so $\mu(M_1'') = \sum_{k=1}^{\infty}(\sum_{i=1}^{2^{k-1}} \mu(W_{q_i}^{k-1}) = \sum_{k=1}^{\infty} \frac{2^k}{6^k} = \frac{1}{2}$. ☐

Corollary 5.5.10. *M_1 is measurable and $\mu(M_1) = 1$.*

Proof. Since M_1' and M_1'' are measurable we have M_1 measurable, so $\mu(M_1) = 1$. ☐

We will now define the representative M_2 of F for which we can not prove M_2 is measurable. We define $M_2 = M_2' \cup M_2''$, where $M_2' = L_2' \cup M''''$ and:

$$\begin{aligned}
M'''' := \{\ x \mid x \in [0,1] \mid \exists l \in \mathbb{N}\Big[& l \leq k_1 \implies \big[x = \sum_{n=1}^{l} \frac{a_n}{6^n} + \frac{2}{6^k}\sum_{m=1}^{\infty}\frac{b_m}{3^m} \wedge \\
& \qquad \forall n < l[a_n \in \{0, \frac{3^n+5}{2}\}] \wedge a_l = \frac{3^k+1}{2} \wedge \\
& \qquad \exists l' \in \mathbb{N} \forall m > l'[b_m \neq 1]\big] \wedge \\
& l > k_1 \implies \big[x = \sum_{n=1}^{k_1} \frac{a_n}{6^n} + \frac{\frac{3_1^k}{2}}{6^{k_1}}\sum_{m=1}^{\infty}\frac{b_m}{3^m} \wedge \\
& \qquad \forall n \leq k_1[a_n \in \{0, \frac{3^n+5}{2}\}] \wedge \\
& \qquad \forall m \leq l[b_m \in \{0,2\}] \wedge b_l = 1 \wedge \\
& \qquad \exists l' \geq l \forall m > l'[b_m \neq 1]\big]\Big\}
\end{aligned}$$

$$\begin{aligned}
M_2'' = \{\ x \mid x \in [0,1] \mid \exists l \in \mathbb{N}\Big[& l \leq k_1 \implies \big[x = \sum_{n=1}^{l} \frac{a_n}{6^n} + \frac{2}{6^k}\sum_{m=1}^{\infty}\frac{b_m}{3^m} \wedge \\
& \qquad \forall n < l[a_n \in \{0, \frac{3^n+5}{2}\}] \wedge a_l = \frac{3^k+1}{2} \wedge \\
& \qquad \forall l' \in \mathbb{N} \exists m > l'[b_m = 1]\big] \wedge \\
& l > k_1 \implies \big[x = \sum_{n=1}^{k_1} \frac{a_n}{6^n} + \frac{\frac{3_1^k}{2}}{6^{k_1}}\sum_{m=1}^{\infty}\frac{b_m}{3^m} \wedge \\
& \qquad \forall n \leq k_1[a_n \in \{0, \frac{3^n+5}{2}\}] \wedge \\
& \qquad \forall m \leq l[b_m \in \{0,2\}] \wedge b_l = 1 \wedge \\
& \qquad \forall l' \geq l \exists m > l'[b_m = 1]\big]\Big\}
\end{aligned}$$

First we will prove that M_2 is a representative of F and then we will show that we can not prove that M_2 is measurable.

Lemma 5.5.11. $M_1 \sim M_2$.

Proof. This will, again, be very similar to the proof of 5.4.5 and we will not give a detailed proof again. □

Lemma 5.5.12. *We can not prove that M_2 is measurable.*

Proof. By lemma 3.3.12 it is enough to prove that M_2' is not measurable. Suppose $\neg\exists n \in \mathbb{N}[n = k_1]$ then $M_2' = M_1'$ so $\mu(M_2') = \frac{1}{2}$. Suppose $\exists n \in \mathbb{N}[n = k_1]$ then $\mu(M_2') = \mu(L_2') + \mu(M'''') = 0+0 = 0$. But $0 < \frac{1}{2}$ so if M_2' is measurable then $\mu(M_2') > 0$ or $\mu(M_2) < \frac{1}{2}$. Suppose $\mu(M_2') > 0$ then $\neg\exists n \in \mathbb{N}[n = k_1]$, but we can not prove this so we can not prove $\mu(M_2') > 0$. Suppose $\mu(M_2') < \frac{1}{2}$ then $\neg\neg\exists n \in \mathbb{N}[n = k_1]$, but we can not prove this so we can not prove $\mu(M_2') > \frac{1}{2}$. This means we can not prove M_2' is measurable. □

Furthermore we can, again, for each representative F' of F find a totally bounded perfect set X such that for each $x \in X$ we can not prove $x \in F'$.

Lemma 5.5.13. *For every representative F' of F we can find a totally bounded perfect set X such that for each $x \in X$ we can not prove $x \in F'$.*

Proof. By lemma 4.10 it is enough to show it for M. Consider

$$X = \{\ x \mid x \in [0,1] \mid x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \ |\forall n > k_1[a_n = 1] \wedge\ \forall n \leq k_1[a_n \in \{0,2\}]\ \}$$

We will first show that X is totally bounded. Pick any $m \in \mathbb{N}$. Suppose $m < k_1$. We define $p_0, p_1, \ldots, p_{2^{m-1}-1}$ with $p_i = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ such that $a_n = 2 * i_n$ for each $n \leq m-1$, $a_n = 0$ for each $m \leq n \leq k_1$ and $a_n = 1$ for each $n > k_1$ where $i_1, \ldots, i_{m-1}$ is the binary notation of i.
Suppose $m \geq k_1$. We define $p_0, p_1, \ldots, p_{2^{k_1}-1}$ with $p_i = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ such that $a_n = 2 * i_n$ for each $n \leq k_1$ and $a_n = 1$ for each $n > k_1$ where $i_1, \ldots, i_{m-1}$ is the binary notation of i.
Now we will prove that X is closed. For this we need to show:

(i) $\forall x \in X\ \exists x' \in \bar{X}$ such that $x \equiv x'$

(ii) $\forall x' \in \bar{X}\ \exists x \in X$ such that $x \equiv x'$

Since $X \subseteq \bar{X}$ (i) is clear. Now take $x' \in \bar{X}$. We will construct $x = \sum_{n=1}^{\infty} \frac{b_n}{3^n} \in X$ by defining a_n for every $n \in \mathbb{N}$. Pick $n \in \mathbb{N}$ and find $y_n = \sum_{k=1}^{\infty} \frac{(a_n)_k}{3^k} \in X$ such that $|y_n - x'| < \frac{1}{3^{n+1}}$. Define $b_n = (a_n)_n$. We will show, for every $n \in \mathbb{N}$, $|x - x'| \leq \frac{1}{3^n}$, so by lemma 1.2.6 $x \equiv x'$. By lemma 1.1.7 it is sufficient to prove $|x - y_{n+1}| \leq \frac{1}{3^{n+1}}$ and $|y_{n+1} - x'| \leq \frac{1}{3^{n+2}}$. By definition of y_{n+1} we already know $|y_{n+1} - x'| \leq \frac{1}{3^{n+2}}$. Thus we will prove $|x - y_{n+1}| \leq \frac{1}{3^{n+1}}$. With the following claim we see it will be sufficient to prove $b_1 = (a_{n+1})_1, \ldots, b_{n+1} = (a_{n+1})_{n+1}$.

Claim. *For* $y = \sum_{n=1}^{\infty} \frac{c_n}{3^n}, y' = \sum_{n=1}^{\infty} \frac{c'_n}{3^n} \in X$ *if* $c_1 = c'_1, \ldots, c_k = c'_k$ *then* $|y' - y| \leq \frac{1}{3^k}$ *and if* $|y' - y| < \frac{1}{3^k}$ *then* $c_1 = c'_1, \ldots, c_k = c'_k$.

Proof. Suppose $y = \sum_{n=1}^{\infty} \frac{c_n}{3^n}, y' = \sum_{n=1}^{\infty} \frac{c'_n}{3^n} \in X$. Suppose $c_1 = c'_1, \ldots, c_k = c'_k$ then $|y - y'| = |\sum_{n=k+1}^{\infty} \frac{c_n}{3^n} - \sum_{n=k+1}^{\infty} \frac{c'_n}{3^n}| \leq \frac{1}{3^k}$. Now suppose $|y' - y| < \frac{1}{3^k}$ and suppose there exists $m \leq k$ such that $c_m \neq c'_m$. Find the smallest $m \leq k$ such that $c_m \neq c'_m$. Decide $m \leq k_1$ or $m > k_1$. Suppose $m \leq k_1$ then $c_m, c'_m \in \{0, 2\}$. Suppose without loss of generality $c_m = 0$ and $c'_m = 2$. Then $y \leq \sum_{n=1}^{m-1} \frac{c_n}{3^n} + \frac{1}{3^m}$ and $y' \geq \sum_{n=1}^{m-1} \frac{c_n}{3^n} + \frac{2}{3^m}$ so $|y - y'| \geq \frac{1}{3} \geq \frac{1}{3^k}$, which is a contradiction. Suppose $m < k_1$, then $c_m = c'_m = 1$, which is a contradiction. So not there exists $m \leq k$ such that $c_m \neq c'_m$ which means $c_m = c'_m$ for all $m \leq k$. □

We will prove $b_1 = (a_{n+1})_1, \ldots b_{n+1} = (a_{n+1})_{n+1}$ with induction, by using the above claim. By definition $b_1 = (a_1)_1$. Now suppose we know $b_1 = (a_n)_1, \ldots, b_n = (a_n)_n$. Then, since $|y_n - x'| \leq \frac{1}{3^{n+1}}$ and $|x' - y_{n+1}| \leq \frac{1}{3^{n+2}}$, by lemma 1.1.7 $|y_n - y_{n+1}| \leq \frac{1}{3^n}$. So, by the above claim, $(a_n)_1 = (a_{n+1})_1, \ldots (a_n)_n = (a_{n+1})_n$, so $b_1 = (a_{n+1})_1, \ldots b_{n+1} = (a_{n+1})_n$ and by definition $b_{n+1} = (a_{n+1})_{n+1}$. □

Moreover, Brouwer again claims we can construct a representative F_1 of F for which we can define a set X such that for each $x \in X$ we can not prove $x \in F_1$ and such that the measure of X is 1. Brouwer here gives a number of definitions of sets which are composed to a set F_1 and a set X. For us it is not clear what he tries to define, so we will not cover this construction in this thesis. Lastly, we will show, for every representative F' of F there exists discontinuous functions $f : F' \to \mathbb{R}$.

Lemma 5.5.14. *For every representative* F' *of* F *there exists discontinuous functions* $f : F' \to \mathbb{R}$.

Proof. By lemma 4.3 it is enough to prove this for M. Now define $f : M \to \mathbb{R}$ with $f(x) = 0$ if $x \in M'$ and $f(x) = 1$ if $x \in M''$. To prove f is discontinuous we consider any $x \in M'$ and $n = 2$. Since $x \in M'$ we can find a sequence $a_1, a_2, a_3, \cdots \in \{0, 1, 2\}$ such that $x = \sum_{n=1}^{infty} \frac{a_n}{3^n}$ and such that there exist $p \in \mathbb{N}$ such that for all $n > p$, $a_n \neq 1$. Pick any $m \in \mathbb{N}$ and find $k \in \mathbb{N}$ such that $\sum_{n=k+1}^{\infty} \frac{2}{3^n} \leq \frac{1}{2^m}$. Define $y = \sum_{n=1}^{\infty} \frac{b_n}{3^n}$ with $b_n = a_n$ for all $n \leq k$ and $b_{k+n} = 1$ for all $n \geq 1$. Then $y \in M''$. We have $|x - y| \leq \sum_{n=k+1}^{\infty} \frac{2}{3^n} \leq \frac{1}{2^m}$ but $|f(x) - f(y)| = 1 > \frac{1}{4}$. □

5.6 Example 7

We define G to be the geometric type of N [(11)], where $N = M' \cup [0, 1] \setminus M'$.

Lemma 5.6.1. N *is measurable and* $\mu(N) = 1$.

[(11)] This is K_0 from example 7 of Brouwers article.

Proof. We already know M' is measurable and $\mu(M') = 0$. This means $[0,1] \setminus M'$ is measurable and $\mu([0,1] \setminus M') = 1$. So N is measurable and $\mu(N) = 1$. □

Lemma 5.6.2. *Every representative of G is of the form $X' \cup X''$ for some $X', X'' \subseteq [0,1]$ such that $X'' = [0,1] \setminus X'$.*

Proof. Suppose G' is a representative of G then $M' \cup ([0,1] \setminus M') \sim G'$. So there exists a uniformly continuous bijection $f : [0,1] \to [0,1]$ such that $f(M' \cup ([0,1] \setminus M')) = E'$. But $f(M' \cup ([0,1] \setminus M')) = f(M') \cup f([0,1] \setminus M')$. Define $X' = f(M')$ and $X'' = f([0,1] \setminus M')$. Now suppose $x \in X''$, then $f^{-1}(x) \in [0,1] \setminus M'$ so $f^{-1}(x) \notin M'$ thus $x \notin X'$. Suppose $x \notin X'$ then $f^{-1}(x) \notin M'$ so $f^{-1}(x) \in [0,1] \setminus M'$ thus $x \in X''$. □

Lemma 5.6.3. *Every measurable representative of G has measure* 1*.*

Proof. Suppose G' is a measurable representative of G, then G' is of the form $X' \cup X''$ for some $X', X'' \subseteq [0,1]$ such that $X'' = [0,1] \setminus X'$. Since $X' \cup X''$ is measurable, by lemma 3.3.10, $(X' \cup X'') \cup ([0,1] \setminus (X' \cup X''))$ is almost full. But $(X' \cup X'') \cup ([0,1] \setminus (X' \cup X'')) = ([0,1] \setminus X') \cup X' \cup \{x \in [0,1] | x \notin [0,1] \setminus X' \text{ and } x \notin X'\} = ([0,1] \setminus X') \cup X'$. So $([0,1] \setminus X') \cup X' = \text{dom}(\chi_{X'})$ is almost full. Also $\chi_{X'}$ is bounded, so by theorem 3.3.4, X' is measurable. Suppose $\mu(X') = k$. By theorem 3.3.15 $[0,1] \setminus X'$ is measurable and $\mu([0,1] \setminus X') = 1 - k$. Also, by part 2. of the proof of theorem 3.3.15, $\mu(X' \cup ([0,1] \setminus X')) = \mu(X' \cup X'') = 1$. □

We will also show that there exists a representative N_2 of G which is not measurable. To define this representative we first have to define the measurable representative N_1.
We define $N_1 = M_1' \cup ([0,1] \setminus M_1')$. This is clearly a representative of G since we already saw $M' \sim M_1'$ so $([0,1] \setminus M') \sim ([0,1] \setminus M_1')$ so $N \sim N_1$. Also, clearly N_1 is measurable since M_1' is measurable with $\mu(M_1') = \frac{1}{2}$.
Now we define $N_2 = M_2' \cup ([0,1] \setminus M_2')$. Again, clearly this is a representative of G since $M_2' \sim M$. Also M_2' is not measurable so $([0,1] \setminus M_2')$ is not measurable, thus by lemma 3.3.12 N_2 is not measurable.

Lemma 5.6.4. *For every representative G' of G we have:*

(i) G' is not apart from $[0,1]$

(ii) G' does not deviate from $[0,1]$

(iii) G' does not coincide with $[0,1]$

Proof. By lemma 4.2 it is enough to prove (i), (ii) and (iii) for N.

(i) Suppose $N \# [0,1]$ then $\exists x \in [0,1][x \# N]$ or $\exists x \in N[x \# [0,1]$. Since $N \subseteq [0,1]$ we must have $\exists x \in [0,1][x \# N]$. Find this x. Then $x \# M'$ so $x \notin M'$ so $x \in ([0,1] \setminus M')$. But $x \# [0,1] \setminus M'$, which is a contradiction. This means $\neg[N \# [0,1]$.

(ii) Suppose $N \not\equiv [0,1]$ then $\exists x \in [0,1] \neg[x \in_0 N]$ or $exists x \in N \neg[x \in_0 [0,1]$. Since $N \subseteq [0,1]$ we must have $\exists x \in [0,1] \neg[x \in_0 L]$. Find this x. Then $\neg x \in_0 M'$ so $x \notin M'$ so $x \in ([0,1] \setminus M')$. But $\neg x \in_0 ([0,1] \setminus M')$, which is a contradiction. This means $\neg[N \not\equiv [0,1]]$.

(iii) This follows directly from lemma 2.2.8

□

Again, Brouwer claims we can construct a representative G_1 of G for which we can define a set X such that for each $x \in X$ we can not prove $x \notin G_1$ and such that X is measurable and $\mu(X) = 1$. To construct this representative he uses the representative F_1. Since for us it is not clear what F_1 is, it is also not clear what G_1 is.
Lastly, we will show, for every representative G' of G there exists discontinuous functions $f : G' \to \mathbb{R}$.

Lemma 5.6.5. *For every representative G' of G there exists discontinuous functions $f : G' \to \mathbb{R}$.*

Proof. By lemma 4.3 it is enough to prove this for N. Now define $f : N \to \mathbb{R}$ with $f(x) = 0$ if $x \in M'$ and $f(x) = 1$ if $x \notin M'$. Since $M'' \subseteq ([0, 1] \setminus M')$ the proof of lemma 5.5.14 also proves that this function is discontinuous. □

6 Conclusion

Brouwers goal was to find a pseudofull subset of $[0, 1]$ which is very much 'alike' $[0, 1]$ such that functions defined on these pseudofull domains are not necessarily (uniformly) continuous. He tries to find a property, which from a classical point of view, means they coincide with $[0, 1]$. In the first example we see that if we have $X \subseteq [0, 1]$ such that X seems to coincide with $[0, 1]$ we can not guarantee that there exists an element in X. This shows that, even with the strongest form of being 'alike', only this property is not enough for a pseudofull subset of $[0, 1]$ and we also need our pseudofull domain to be almost full. By the continuity principle we can even conclude that it is too much to state that these pseudofull domains should seem to coincide with $[0, 1]$, since then we exclude all sets which are the union of two disjoint sets which both contain at least one element. This is why Brouwer concludes that the properties for being 'alike' $[0, 1]$ should be that a pseudofull domain does not deviate from $[0, 1]$.
We also see in example 2 that only the property of being almost full is not enough, since every representative of B is apart from $[0, 1]$ and function defined on any representative of B are continuous. This why Brouwer concludes that the pseudofull subsets of $[0, 1]$ should not deviate from $[0, 1]$ and should be almost full.

References

[1] L.E.J. Brouwer, *Over de grondslagen der wiskunde.* Thesis, Amsterdam, 1907. (Collected Works Volume 1,North-Holland Pu. Com., Amsterdam, 1975, p 11)

[2] L.E.J. Brouwer, *Begrüngung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossen Dritten. Zweiter Teil: Theorie der Punktmengen.* KNAW Verhandelingen, 1^{e} sectie 12 no. 7, 1919.
(Collected Works Volume 1,North-Holland Pu. Com., Amsterdam, 1975, p 191)

[3] L.E.J. Brouwer, *Intuitionistische Zerlegung mathematischer Grundbegriffe.* Jber.Deutsch.Math.Verein.33 (1924), p 251-256, Amsterdam, 1923.
(Collected Works Volume 1,North-Holland Pu. Com., Amsterdam, 1975, p 275)

[4] L.E.J. Brouwer, *Über Definitionsbereiche von Funktionen.* Math. Analen 97, p. 60-75, Amsterdam, 1927.
(Collected Works Volume 1,North-Holland Pu. Com., Amsterdam, 1975, p 390)

[5] L.E.J. Brouwer, *Intuitionistische Betrachtungen über den Formalismus.* KNAW Verhandelingen, 31, p 374-379, 1928.
(Collected Works Volume 1,North-Holland Pu. Com., Amsterdam, 1975, p 409)

[6] A. Heyting, *Intuitionism, An introduction.* North-Holland Pu. Com., second revised edition, Amsterdam, 1966.

[7] S.C. Kleene, R.E. Vesley, *The foundations of intuitionistic mathematics, especially in relation to the theory of recursive functions.* North-Holland Pu. Co., Amsterdam, 1965

[8] P. Mancosu, *From Brouwer To Hilbert, The Debate on the Foundations of Mathematics in the 1920s.* Part IV, section 18, Oxford University Press, Oxford, 1998.

[9] Wim Veldman, *Brouwer's Fan Theorem as an axiom and as a contrast to Kleene's Alternative.* Springer-Verlag, Archive for Mathematical Logic, 53, 5-6, pp. 621-693, New York, 2014.

[10] Wim Veldman, *Intuïtionistische Wiskunde.* Instituut WiNSt, Radboud Universiteit Nijmegen, 2008.