# Mathematics
# ANSWERS Test Problems

## Problem 1

Solve the following equation:

$$\frac{4}{x+3} + \frac{3x}{x-4} - \frac{x-1}{x^2-x-12} = 0$$

**Answer**

Because $x^2 - x - 12 = (x+3)(x-4)$ we have

$$\frac{4}{x+3} + \frac{3x}{x-4} - \frac{x-1}{x^2-x-12} = \frac{4(x-4)+3x(x+3)-(x-1)}{(x+3)(x-4)} = \frac{3x^2+12x-15}{(x+3)(x-4)}$$

This last expression is equal to 0 if the numerator equals 0 and $x \neq$ -3 or 4. We have

$$3x^2 + 12x - 15 = 3(x-1)(x+5) = 0$$

Therefore, $x = 1$ and $x = -5$ solve the equation.

## Problem 2

Consider the expression below. Determine all values of $x$ for which this expression is positive, 0 and negative, respectively. (*Recall that Euler's number* $e \approx 2.718$):

$$-(x-1)^2 \frac{x^2-2}{2e^x-4}$$

**Answer**

We split the expression in three parts that we first consider seperately.

$-(x-1)^2$: Note that $(x-1)^2 \geq 0$ for all $x$; therefore, $-(x-1)^2 \leq 0$ for all $x$. In particular, $-(x-1)^2 = 0$ only if $x = 1$. Hence,

$$-(x-1)^2 \begin{cases} = 0 & \text{if } x = 1 \\ < 0 & \text{if } x \neq 1 \end{cases}$$

$x^2 - 2$: We see

$$x^2 - 2 \begin{cases} > 0 & \text{if } x < -\sqrt{2} \text{ or } x > \sqrt{2} \\ = 0 & \text{if } x = -\sqrt{2} \text{ or } x = \sqrt{2} \\ < 0 & \text{if } -\sqrt{2} < x < \sqrt{2} \end{cases}$$

$\frac{1}{2e^x-4}$: Note that $2e^x - 4 > 0$ if $e^x > \frac{4}{2}$ , which means $x > \ln(2)$; $2e^x - 4 = 0$ if $x = \ln(2)$ and $2e^x - 4 < 0$ if $x < \ln(2)$. From this it follows that $\frac{1}{2e^x-4}$ is not defined for $x = \ln(2)$, and that

$$\frac{1}{2e^x-4} \begin{cases} > 0 & \text{if } x > \ln(2) \\ < 0 & \text{if } x < \ln(2) \end{cases}$$

Note that $-\sqrt{2} < 0 = \ln(1) < \ln(2)$. Because $2 < e$ it follows that $\ln(2) < \ln(e) = 1$. Furthermore, $1 = \sqrt{1} < \sqrt{2}$. To summarize: $-\sqrt{2} < \ln(2) < 1 < \sqrt{2}$. The diagram below shows where the different parts are positive (+), zero (0), negative (-) or undefined (N). The last line shows the sign of the whole expression.

| | | $-\sqrt{2}$ | | $\ln(2)$ | | 1 | | $\sqrt{2}$ | |
|---|---|---|---|---|---|---|---|---|---|
| $-(x-1)^2$ | - | - | - | - | - | 0 | - | - | - |
| $x^2-2$ | + | 0 | - | - | - | - | - | 0 | + |
| $\frac{1}{2e^x-4}$ | - | - | - | N | + | + | + | + | + |
| $-(x-1)^2 \frac{x^-2}{2e^x-4}$ | + | 0 | - | N | + | 0 | + | 0 | - |

## Problem 3

Solve the system of equations (1)-(3) for $x$, $y$ and $z$:

$$x^2 + 4y - 4z - zx = 8 \quad (1)$$

$$y - x - 2z = 0 \quad (2)$$

$$e^{x-y} = \frac{1}{e^2} \quad (3)$$

**Answer**

From (3) it follows that:

$$\begin{aligned} e^{x-y} &= \tfrac{1}{e^2} \\ \Rightarrow \quad x - y &= -2 \end{aligned}$$

When we substitute $x - y = -2$ in (2) we obtain:

$$\begin{aligned} y - x - 2z &= 0 \\ \Rightarrow \quad -2 - 2z &= 0 \\ \Rightarrow \quad z &= 1 \end{aligned}$$

When we substitute $y = x + 2$ and $z = 1$ in (1), it follows that:

$$\begin{aligned} x^2 + 4y - 4z - zx &= 8 \\ \Rightarrow \quad x^2 + 4(x+2) - 4 - x &= 8 \\ \Rightarrow \quad x^2 + 3x - 4 &= 0 \\ \Rightarrow \quad (x-1)(x+4) &= 0 \end{aligned}$$

This means that $x = 1$ or $x = -4$. We conclude that the system has two solutions: $(x, y, z) = (1, 3, 1)$ and $(x, y, z) = (-4, -2, 1)$.

## Problem 4

Solve the following equation (*Recall that $|u|$ denotes the absolute value of $u$*):

$$\sum_{i=2}^{6} i^{-1}x^2 = \frac{9}{20}x^2 + \sum_{i=1}^{5} x^{|i-3|}$$

**Answer**

We write down the terms of the summation explicitly and solve the equation:

$$\begin{array}{rrcl}
 & \sum_{i=2}^{6} i^{-1}x^2 & = & \frac{9}{20}x^2 + \sum_{i=1}^{5} x^{|i-3|} \\
\Rightarrow & \left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)x^2 & = & \frac{9}{20}x^2 + x^2 + x + 1 + x + x^2 \\
\Rightarrow & \left(\frac{30}{60}+\frac{20}{60}+\frac{15}{60}+\frac{12}{60}+\frac{10}{60}\right)x^2 - \frac{27}{60}x^2 & = & 2x^2 + 2x + 1 \\
\Rightarrow & x^2 & = & 2x^2 + 2x + 1 \\
\Rightarrow & x^2 + 2x + 1 & = & 0 \\
\Rightarrow & (x+1)(x+1) & = & 0
\end{array}$$

It follows that $x = -1$.

## Problem 5

Solve the following equation:

$$2^{6\ln(x)} = x^{3\ln(x)}$$

**Answer**

We derive:

$$\begin{array}{rrcl} & 2^{6\ln(x)} & = & x^{3\ln(x)} \\ \Rightarrow & \ln\left(2^{6\ln(x)}\right) & = & \ln\left(x^{3\ln(x)}\right) \\ \Rightarrow & 6\ln(x)\ln(2) & = & 3\ln(x)\ln(x) \\ \Rightarrow & \ln(x)\left(2\ln(2) - \ln(x)\right) & = & 0 \end{array}$$

The solution of this equation is given by $\ln(x) = 0$ or $\ln(x) = 2\ln(2) = \ln(2^2) = \ln(4)$; hence, $x = 1$ and $x = 4$.

## Problem 6

Solve the system of equations (4)-(6) for $x$, $y$ and $z$:

$$z^2 - x^2 - 2xy = 1 \tag{4}$$

$$\ln\left(\frac{x+y}{z}\right) = 0 \tag{5}$$

$$x^2 - 2y^2x - 4y^2 = 4 \tag{6}$$

**Answer**

Note from (5) that $z \neq 0$. Besides, from (5) it follows that:

$$\begin{aligned} \ln\left(\tfrac{x+y}{z}\right) &= 0 \\ \Rightarrow \quad \tfrac{x+y}{z} &= e^0 \\ \Rightarrow \quad x+y &= z \end{aligned}$$

If we substitute $z = x + y$ in (4) it follows that:

$$\begin{aligned} z^2 - x^2 - 2xy &= 1 \\ \Rightarrow \quad (x+y)^2 - x^2 - 2xy &= 1 \\ \Rightarrow \quad y^2 &= 1 \end{aligned}$$

If we substitute $y^2 = 1$ in (6) it follows that:

$$\begin{aligned} x^2 - 2y^2x - 4y^2 &= 4 \\ \Rightarrow \quad x^2 - 2x - 4 &= 4 \\ \Rightarrow \quad x^2 - 2x - 8 &= 0 \\ \Rightarrow \quad (x-4)(x+2) &= 0 \end{aligned}$$

That is, $x = 4$ or $x = -2$. The four solutions of this system of equations are $(x, y, z) = (4, 1, 5)$, $(x, y, z) = (4, -1, 3)$, $(x, y, z) = (-2, 1, -1)$ and $(x, y, z) = (-2, -1, -3)$.